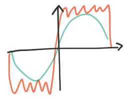
# Multivariable Calculus 

Lectured in Autumn 2015 at Imperial College by Dr. A. G. Walton.
Humbly typed by Karim Bacchus.

Caveat Lector: unofficial notes. Comments and corrections should be sent to kb514@ic.ac.uk. Other notes available at wwwf.imperial.ac.uk/∼kb514.

## Syllabus

*The calculus of functions of several independent variables is developed in this module, together with an introduction to partial differential equations.*

*Vector Calculus:* Preliminary ideas and some revision of vectors: tensor notation and the summation convention. Gradient: components of grad in different coordinate systems, tangent planes, divergence and curl, operations with the gradient, the Laplacian, scalar and vector fields. Line integrals: conservative forces, circulation. Surface integrals: definition, projection theorem. Volume integrals: definition. Results relating line, surface and volume integrals: Greens theorem, flux, the divergence theorem, Gauss theorem, Stokes theorem. Curvilinear coordinates: line and volume elements, gradient, divergence, curl, Laplacian. Changes of variable in surface integration: Jacobian.

*Fourier series:* orthonormal systems, periodic functions, even and odd functions, fullrange series, the Gibbs phenomenon, Parsevals theorem, half-range series, integration and differentiation of Fourier series, exponential form.

*Fourier transforms:* exponential, cosine and sine transforms, elementary properties, convolution theorem, energy theorem, Dirac delta function.

*Partial differential equations:* the wave equation: method of separation of variables, use of Fourier transforms, D'Alembert's solution; the heat equation: solution using same methods as above; Laplace and Poisson equation: types of boundary conditions, uniqueness of solution, point sources, Greens functions, method of images.

## Appropriate books

Spiegel, *Vector Analysis*, Schaums Outline Series.

R. Courant, *Differential and Integral Calculus Vol II.*

R. Haberman, *Elementary applied partial differential equations with Fourier series and boundary value problems*

E. Kreyszig, *Advanced Engineering Mathematics*

# Contents

# 1 Vector Calculus

## 1.1 Preliminary Ideas and some revision of vectors

### Einstein summation convention

In any product of terms, if we have a repeated suffix, then that quantity is considered to be summed over (from 1 to 3, since we will usually be working in three dimensions). For example $a_i x_i$ is shorthand for $\sum_{i=1}^{3} a_i x_i$. Lecture 1

**Definition** (The Kronecker delta)**.** This is the quantity $\delta_{ij}$ and is defined such that

$$\delta_{ij} = \begin{cases} 1, i = j; \\ 0, i \neq j \end{cases}$$

E.g. $\delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = a_i$.

Note that the left-hand side had two different subscripts, while the right-hand-side ends up with only one subscript - this is known as a *contraction.*

**Definition** (The permutation symbol)**.** This is the quantity $\varepsilon_{ijk}$ defined as

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two of } i, j, k \text{ are the same;} \\ 1, & \text{if } i, j, k \text{ is an } \textit{even} \text{ permutation of } 1, 2, 3; \\ -1, & \text{if } i, j, k \text{ is an } \textit{odd} \text{ permutation of } 1, 2, 3 \end{cases}$$

An even permutation is a cyclic permutation, i.e. $(123), (231), (312)$. An odd permutation is an acyclic permutation, i.e. $(321), (213), (132)$. For example $\varepsilon_{123} = 1$, $\varepsilon_{321} = -1$, $\varepsilon_{133} = 0$.

We can show, by considering the various cases, that the Kronecker delta and the permutation symbol are connected by the formula

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

The quantities $\delta_{ij}$ and $\varepsilon_{ijk}$ are known as *tensors.*

**Definition** (Vector Product)**.** This is the multiplication of two vectors which results in a third vector, perpendicular to the first two. It can be written in the form of a determinant as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If $\mathbf{a}\times\mathbf{b}=0$, then the two vectors are parallel. Recall that $(\mathbf{a}\times\mathbf{b})=-(\mathbf{b}\times\mathbf{a})$. If we just consider the first component of this vector we can write this as

$$\begin{aligned} a_2b_3-a_3b_2 &= \varepsilon_{123}a_2b_3+\varepsilon_{132}a_3b_2 \\ &= \varepsilon_{1jk}a_jb_k \\ &= \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{1jk}a_jb_k \end{aligned}$$

Since $\varepsilon_{123}=1, \varepsilon_{132}=-1$, and $\varepsilon_{1ij}=0$ for all other $i$ and $j$. In general we can write the $i$th component of $a\times b$ as

$$[a\times b]_i=\varepsilon_{ijk}a_jb_k$$

> **Definition** (Scalar Product)**.** This is defined as
>
> $$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1b_1+a_2b_2+a_3b_3 \\ &= a_ib_i \end{aligned}$$
>
> using the summation convention.

Recall that if $\mathbf{a}\cdot\mathbf{b}=0$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

> **Definition** (Triple scalar product)**.** This is the quantity
>
> $$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= a_i[\mathbf{b}\times\mathbf{c}]_i \\ &= a_i\varepsilon_{ijk}b_jc_k \\ &= \varepsilon_{ijk}a_ib_jc_k \end{aligned}$$

If this quantity is zero then the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar. A useful property of the triple scalar prodcut is that the dot and cross can be swapped without changing the answer provided the order of the vectors remained unchanged, i.e. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, as $\varepsilon_{ijk}a_ib_jc_k=(\varepsilon_{kij}a_ib_j)c_k=[\mathbf{a}\times\mathbf{b}]_kc_k$.

> **Definition** (Triple vector product)**.** This is defined as
>
> $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

Since $\mathbf{b}\times\mathbf{c}$ is a vector normal to the plane of $\mathbf{b}$ and $\mathbf{c}$, and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is normal to $\mathbf{b}\times\mathbf{c}$, it follows that the triple vector product must lie in the plane of $\mathbf{b}$ and $\mathbf{c}$. In component notation

$$\begin{aligned} [\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i &= \varepsilon_{ijk}a_j[\mathbf{b}\times\mathbf{c}]_k \\ &= \varepsilon_{ijk}a_j\varepsilon_{klm}b_lc_m \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_jb_lc_m \\ &= a_jb_ic_j-a_jb_jc_i \\ &= (\mathbf{a}\cdot\mathbf{c})b_i-(\mathbf{a}\cdot\mathbf{b})c_i \end{aligned}$$

and so we conclude that

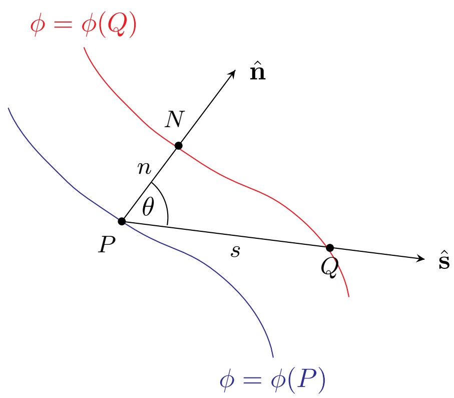
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

which confirms explicitly that the triple vector product does lie in the plane of $\mathbf{b}$ and $\mathbf{c}$.

## 1.2 Gradient

Let $\phi$ be a differentiable scalar function of position in three dimensions. If $P$ is a general point, $\phi$ will depend on the position of $P$, so we may write $\phi = \phi(P)$. The position of $P$ is defined by the reference to a coordinate system e.g. if we consider Cartesian coordinates, coordinates, then $P$ depends on $(x, y, z)$ and hence $\phi = \phi(x, y, z)$, while if we consider cylindrical polar coordinates $(r, \theta, z)$ then $\phi = \phi(r, \theta, z)$. Lecture 2

The equation $\phi$ = constant defines a surface in three dimensions. Varying the constant, we can define a family of surfaces called *level surfaces* or *equi-$\phi$ surfaces*. For example, if $\phi$ represents pressure, then $\phi$ = constant defines a family of surfaces over which the pressure is constant. The surface through a **specific point** $P$ is $\phi = \phi(P)$. Let $Q$ be a neighbouring point. (See picture).




The equation of the level surface through $Q$ is $\phi(Q)$. We draw the normal to $\phi = \phi(P)$ at $P$. Suppose that it intersects $\phi = \phi(Q)$ at point $N$. Since $N$ is on $\phi = \phi(Q)$, we have $\phi(N) = \phi(Q)$. Let $s$ denote the length along $PQ$ and let $n$ denote the length along $PN$. Introduce unit vectors $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ in those directions.

> **Definition** (Directional derivative)**.** We define
>
> $$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial n}(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}})$$
>
> to be the *directional derivative* of $\phi$ in the direction of $\hat{\mathbf{s}}$.

We derive this:

$$\begin{aligned}
\frac{\partial\phi}{\partial s} &= \lim_{PQ\to 0}(\phi(Q)-\phi(P))/PQ \\
&= \lim_{P\to Q}(\phi(Q)-\phi(P))/PN\cdot(PN/PQ) \\
&= \lim_{N\to P}(\phi(N)-\phi(P))/PN\cdot\lim_{Q\to P}PN/PQ \\
&= \frac{\partial\phi}{\partial n}\cos\theta \\
&= \frac{\partial\phi}{\partial n}(\,\hat{\mathbf{n}}\cdot\hat{\mathbf{s}})
\end{aligned}$$

Since $\cos\theta \le 1$, the maximum directional derivative at $P$ occurs along the normal to $\phi = \phi(P)$ at $P$.

> **Definition.** The vector $\hat{\mathbf{n}}\partial\phi/\partial n$ is the *gradient* of $\phi$ at $P$, written as $\operatorname{grad}\phi$ or $\nabla\phi$. The operator grad or $\nabla$ is known as the *vector gradient operator*.

So we have the directional derivative as

$$\frac{\partial\phi}{\partial s} = \hat{\mathbf{s}}\cdot\nabla\phi$$

## Cartesian Components of $\nabla\phi$

If $\nabla\phi = A_1\,\hat{\mathbf{i}}+A_2\,\hat{\mathbf{j}}+A_3\,\hat{\mathbf{k}}$, then $\hat{\mathbf{i}}\cdot\nabla\phi = A_1$. But, by definition, $\hat{\mathbf{i}}\cdot\nabla\phi = \partial\phi/\partial x$. Hence $A_1 = \partial\phi/\partial x$. Similarly we find $A_2 = \partial\phi/\partial y$, $A_3 = \partial\phi/\partial z$ and so we have the result:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{k}}$$

> **Example 1.1.** *If $\phi = axy^2 + byz + cx^3z^2$, where $a, b, c$ are constant, find $\nabla\phi$. Also find the directional derivative of $\phi$ at the point $(1,4,2)$ in the direction towards the point $(2,0,-1)$.*
>
> $$\nabla\phi = \,\hat{\mathbf{i}}(ay^2 + 3cx^2z^2) + \,\hat{\mathbf{j}}(2axy + bz) + \,\hat{\mathbf{k}}(by + 2cx^3z)$$
>
> At $P = (1,4,2)$ we have
>
> $$(\nabla\phi)_P = \,\hat{\mathbf{i}}(16a + 12c) + \,\hat{\mathbf{j}}(8a + 2b) + \,\hat{\mathbf{k}}(4b + 4c)$$
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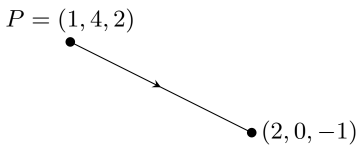
> 
> 

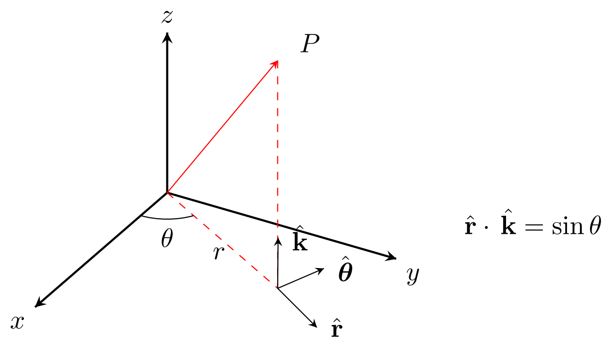
$$\mathbf{s} = (2,0,-1) - (1,4,2) = (1,-4,-3)$$
$$\implies \hat{\mathbf{s}} = \frac{(1,4,-3)}{\sqrt{1^2+4^2+3^2}} = \frac{\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{26}}$$

So the directional derivative is

$$\left(\frac{\partial \phi}{\partial s}\right)_P = (\nabla\phi \cdot \hat{\mathbf{s}})_P$$
$$= \frac{(16a+12c) - 4(8a+2b) - 3(4b+4c)}{\sqrt{26}}$$

## Cylindrical polar components of $\nabla\phi$

The set-up is as shown below.




We write $\nabla\phi = A_1\,\hat{\mathbf{r}} + A_2\,\hat{\boldsymbol{\theta}} + A_3\,\hat{\mathbf{k}}$. Then it follows that

$$A_1 = \hat{\mathbf{r}} \cdot \nabla\phi$$
$$= \hat{\mathbf{r}} \cdot \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}\right)$$
$$= \cos\theta\frac{\partial \phi}{\partial x} + \sin\theta\frac{\partial \phi}{\partial y} + 0$$

We're in plane polar coordinates, so $x = r\cos\theta$ and $y = r\sin\theta$, thus

$$A_1 = \frac{\partial x}{\partial r}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \phi}{\partial y}$$
$$= \frac{\partial \phi}{\partial r} \quad \text{(transformation rule)}$$

Similarly, we find $A_2 = \hat{\boldsymbol{\theta}} \cdot \nabla\phi = (1/r)\partial\phi/\partial\theta$, $A_3 = \hat{\mathbf{k}} \cdot \nabla_\phi = \partial\phi/\partial z$, and hence

$$\nabla\phi = \hat{\mathbf{r}}\frac{\partial \phi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial \phi}{\partial \theta} + \hat{\mathbf{k}}\frac{\partial \phi}{\partial z}$$

### Equation of a tangent plane to $\phi = \phi(P)$

We have that $(\nabla\phi)_P$ is normal to $\phi = \phi(P)$ at $P$. The equation of the tangent plane is therefore

$$(\mathbf{r} - \mathbf{r}_P) \cdot (\nabla\phi)_P = 0$$

ie.

$$\left(\frac{\partial\phi}{\partial x}\right)_P (x - x_P) + \left(\frac{\partial\phi}{\partial y}\right)_P (y - y_P) + \left(\frac{\partial\phi}{\partial z}\right)_P (z - z_P) = 0$$

**Example 1.2.** *Find the tangent plane to the surface $z = e^{-(x^2+y^2)^{1/2}}$ at the point $x = -1, y = 0$.*

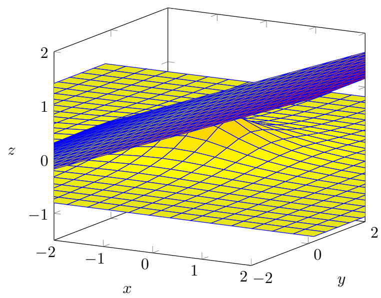
Let $\phi = z - e^{-(x^2+y^2)^{1/2}} = 0$.

$$\begin{aligned} \implies \frac{\partial\phi}{\partial x} &= \frac{xe^{-(x^2+y^2)^{1/2}}}{(x^2+y^2)^{1/2}} \\ &= -e^{-1} \text{ at } (-1, 0) \end{aligned}$$

Similarly $\partial\phi/\partial y = 0$ at $(-1, 0)$ and $\partial\phi/\partial z = 1$ at $(-1, 0)$, and $z_P = e^{-1} \implies$ the tangent plane is

$$\begin{aligned} &-e^{-1}(x - (-1)) + 0 + (1)(z - e^{-1}) = 0 \\ &\implies z = \frac{1}{e}(2 + x) \end{aligned}$$

Which looks like this:




## 1.3 Curl and Divergence

**Definition.** Since $\nabla$ is a vector operator, we can define formally a scalar product $\nabla \cdot \mathbf{A}$. This is called the *divergence* of the vector $\mathbf{A}$. We can also define the *vector product* $\nabla \times \mathbf{A}$, which is called the *curl* of $\mathbf{A}$. Lecture 3

So to summarise, we have

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A}, \quad \operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A}$$

## Cartesian Form

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(A_1\,\hat{\mathbf{i}} + A_2\,\hat{\mathbf{j}} + A_3\,\hat{\mathbf{k}}\right)\\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \longleftarrow \text{ A scalar}\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}\\ &= \hat{\mathbf{i}}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{\mathbf{k}}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \longleftarrow \text{ A vector}\end{aligned}$$

Note that these simple forms for div and curl arise because $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are *constant* vectors: this is not so in other coordinate systems.

**Examples 1.3.**

(a) If

$$A = (y^2\cos x + z^3)\,\hat{\mathbf{i}} + (2y\sin x - 4)\,\hat{\mathbf{j}} + (3xz^2 + 2)\,\hat{\mathbf{k}}$$

*Find* div $\mathbf{A}$ *and* curl $\mathbf{A}$.

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \frac{\partial}{\partial x}(y^2\cos x + z^3) + \frac{\partial}{\partial y}(2y\sin x - 4) + \frac{\partial}{\partial z}(3xz^2 + 2)\\ &= -y^2\sin x + 2\sin x + 6xz\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2\end{vmatrix}\\ &= \hat{\mathbf{i}}(0) - \hat{\mathbf{j}}(3z^2 - 3z^2) + \hat{\mathbf{k}}(2y\cos x - 2y\cos x) = \mathbf{0}\end{aligned}$$

(b) *Find* div $\mathbf{u}$ *and* curl $\mathbf{u}$ *when (i)* $\mathbf{u} = \mathbf{r}$*; (ii)* $\mathbf{u} = \omega\times\mathbf{r}$*, where* $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$*, and* $\omega = \Omega\,\hat{\mathbf{k}}$ *with* $\Omega$ *constant.*

(i) We calculate

$$\operatorname{div}\mathbf{u} = 1 + 1 + 1 = 3$$

$$\operatorname{curl}\mathbf{u} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ x & y & z\end{vmatrix} = \mathbf{0}$$

(ii)

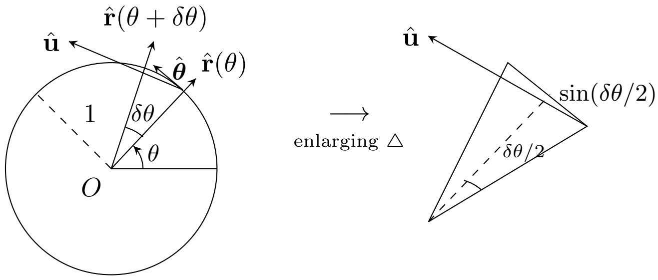
$$\mathbf{u} = \omega \times \mathbf{r}(\omega = \Omega\,\hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix}$$
$$= \hat{\mathbf{i}}(-\Omega y) - \hat{\mathbf{j}}(-\Omega x)$$
$$= -\Omega y\,\hat{\mathbf{i}} + \Omega x\,\hat{\mathbf{j}}$$
$$\implies \operatorname{div}\mathbf{u} = 0 \quad (\mathbf{u} \text{ is } \textit{solenoidal})$$
$$\operatorname{curl}\mathbf{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -\Omega y & \Omega x & 0 \end{vmatrix}$$
$$= \hat{\mathbf{k}}(\Omega + \Omega)$$
$$= 2\Omega\,\hat{\mathbf{k}} = 2\omega$$

Note that the curl $\mathbf{u}$ is related to *rotation.*

## Divergence in cylindrical polar coordinates

$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \nabla\cdot\mathbf{A} \\
&= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot(A_1\,\hat{\mathbf{r}} + A_2\,\hat{\boldsymbol{\theta}} + A_3\,\hat{\mathbf{k}}) \\
&= \hat{\mathbf{r}}\cdot\left(\frac{\partial}{\partial r}(A_1\,\hat{\mathbf{r}}) + \frac{\partial}{\partial r}(A_2\,\hat{\boldsymbol{\theta}}) + \frac{\partial}{\partial r}(A_3\,\hat{\mathbf{k}})\right) \\
&+ \frac{\hat{\boldsymbol{\theta}}}{r}\cdot\left(\frac{\partial}{\partial \theta}(A_1\,\hat{\mathbf{r}}) + \frac{\partial}{\partial \theta}(A_2\,\hat{\boldsymbol{\theta}}) + \frac{\partial}{\partial \theta}(A_3\,\hat{\mathbf{k}})\right) \\
&+ \hat{\mathbf{k}}\cdot\left(\frac{\partial}{\partial z}(A_1\,\hat{\mathbf{r}}) + \frac{\partial}{\partial z}(A_2\,\hat{\boldsymbol{\theta}}) + \frac{\partial}{\partial z}(A_3\,\hat{\mathbf{k}})\right)
\end{aligned}$$

Note how the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary:




Now if we only vary $r$ then the unit vectors stay constant and we have

$$\frac{\partial}{\partial r}(\,\hat{\mathbf{r}},\,\hat{\boldsymbol{\theta}},\,\hat{\mathbf{k}}) = (0,0,0)$$

Similarly

$$\frac{\partial}{\partial z}(\,\hat{\mathbf{r}},\,\hat{\boldsymbol{\theta}},\,\hat{\mathbf{k}}) = (0,0,0)$$

However, if we look at the picture above, we see

$$\begin{aligned}\frac{\partial\,\hat{\mathbf{r}}}{\partial\theta} &= \lim_{\delta\theta\to 0}\frac{\hat{\mathbf{r}}(\theta+\delta\theta)-\,\hat{\mathbf{r}}(\theta)}{\delta\theta}\\ &= \lim_{\delta\theta\to 0}\left[\frac{2\sin(\frac{1}{2}\delta\theta)\,\hat{\mathbf{u}}}{\delta\theta}\right]\\ &= \hat{\boldsymbol{\theta}}\end{aligned}$$

Similarly

$$\frac{\partial\,\hat{\boldsymbol{\theta}}}{\partial\theta} = -\,\hat{\mathbf{r}}$$

(try this as an exercise) and hence

$$\frac{\partial}{\partial\theta}(\,\hat{\mathbf{r}},\,\hat{\boldsymbol{\theta}},\,\hat{\mathbf{k}}) = (\,\hat{\boldsymbol{\theta}},-\,\hat{\mathbf{r}},0)$$

Going back to our expression for div **A** above, we find that

$$\operatorname{div}\mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial\theta} + \frac{\partial A_3}{\partial z}$$

**Example 1.4.** A vector field has the form

$$\mathbf{u} = r\sin(2\theta)\,\hat{\mathbf{r}} + r\cos(2\theta)\,\hat{\boldsymbol{\theta}}$$

*Find the divergence of* **u**.

We calculate

$$\begin{aligned}u_1 &= r\sin 2\theta\\ u_2 &= r\cos 2\theta\\ u_3 &= 0\end{aligned}$$

Then

$$\begin{aligned}\operatorname{div}\mathbf{u} &= \sin 2\theta + \sin 2\theta + \frac{1}{r}(-2r\sin 2\theta)\\ &= \mathbf{0} \quad \text{(solenoidal)}\end{aligned}$$

## 1.4 Operations with the gradient operator

### Important sum and product formulae

Note that $\nabla$ is a linear operator, and so: Lecture 4

(i) $\nabla(\phi_1+\phi_2) = \nabla_{\phi_1} + \nabla_{\phi_2}$.

(ii) $\operatorname{div}(\mathbf{A}+\mathbf{B}) = \operatorname{div}\mathbf{A} + \operatorname{div}\mathbf{B}$.

(iii) $\operatorname{curl}(\mathbf{A}+\mathbf{B}) = \operatorname{curl}\mathbf{A} + \operatorname{curl}\mathbf{B}$.

The proofs of these results follow immediately from the definition of $\nabla$. Other key results are:

(iv) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

(v) $\operatorname{div}(\phi\mathbf{A}) = \phi \operatorname{div}\mathbf{A} + \nabla\phi \cdot \mathbf{A}$.

*Proof of (v).*

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{A}) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot (\phi A_1\,\hat{\mathbf{i}} + \phi A_2\,\hat{\mathbf{j}} + \phi A_3\,\hat{\mathbf{k}}) \\
&= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3) \\
&= \underbrace{\phi\left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right]}_{\phi \operatorname{div}\mathbf{A}} + \underbrace{A_1\frac{\partial \phi}{\partial x} + A_2\frac{\partial \phi}{\partial y} + A_3\frac{\partial \phi}{\partial z}}_{(\mathbf{A}\cdot\nabla)\phi}
\end{aligned}$$ ■

In writing out these proofs it is easier to use the *summation convention* that we introduced earlier. Rather than write $(x, y, z)$ for Cartesian components, we write $(x_1, x_2, x_3)$ and in place of $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, we write $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. Then we saw ealier that

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= A_i B_i, \\
\mathbf{A} \times \mathbf{B} &= \varepsilon_{ijk}\,\hat{\mathbf{e}}_i A_j B_k
\end{aligned}$$

Also recall the useful result that

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Thus, under the summation convention:

$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \frac{\partial A_i}{\partial x_i} \\
[\nabla\phi]_i &= \frac{\partial \phi}{\partial x_i} \\
[\operatorname{curl}\mathbf{A}]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}A_k
\end{aligned}$$

where $[\,]_i$ indicates the $i$th component.

Using this approach, the proof of (v) takes the form:

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{A}) &= \frac{\partial}{\partial x_i}(\phi A_i) \\
&= \phi\frac{\partial A_i}{\partial x_i} + A_i\frac{\partial \phi}{\partial x_i} \\
&= \phi \operatorname{div}\mathbf{A} + (\mathbf{A}\cdot\nabla)\phi
\end{aligned}$$

Other important results are:

(vi) $\operatorname{curl}(\phi\mathbf{A}) = \phi \operatorname{curl}\mathbf{A} + \nabla\phi \times \mathbf{A}$.

(vii) $\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl}\mathbf{A} - \mathbf{A} \cdot \operatorname{curl}\mathbf{B}$.

(viii) $\operatorname{curl}(\mathrm{A} \times \mathrm{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} = \mathbf{B}\operatorname{div}\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\operatorname{div}\mathbf{B}$.

(ix) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B} \times \operatorname{curl}\mathbf{A} + \mathbf{A} \times \operatorname{curl}\mathbf{B}$.

**Example 1.5.** Prove relation (ix) above.

*Proof.*

$$
\begin{aligned}
&[(\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\operatorname{curl}\mathbf{A} + \mathbf{A}\times\operatorname{curl}\mathbf{B}]_i \\
&= B_j\frac{\partial A_i}{\partial x_j} + A_j\frac{\partial B_i}{\partial x_j} + \varepsilon_{ijk}B_j(\operatorname{curl}\mathbf{A})_k + \varepsilon_{ijk}A_j(\operatorname{curl}\mathbf{B})_k \\
&= B_j\frac{\partial A_i}{\partial x_j} + A_j\frac{\partial B_i}{\partial x_j} + \varepsilon_{ijk}\left[B_j\varepsilon_{klm}\frac{\partial A_m}{\partial x_l} + A_j\varepsilon_{klm}\frac{\partial B_m}{\partial x_l}\right] \\
&= B_j\frac{\partial A_i}{\partial x_j} + A_j\frac{\partial B_i}{\partial x_j} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left[B_j\frac{\partial A_m}{\partial x_l} + A_j\frac{\partial B_m}{\partial x_l}\right] \\
&= \cancel{B_j\frac{\partial A_i}{\partial x_j}} + \cancel{A_j\frac{\partial B_i}{\partial x_j}} + B_j\left(\frac{\partial A_j}{\partial x_i} - \cancel{\frac{\partial A_i}{\partial x_j}}\right) + A_j\left(\frac{\partial B_j}{\partial x_i} - \cancel{\frac{\partial B_i}{\partial x_j}}\right) \\
&= B_j\frac{\partial A_i}{\partial x_i} + A_i\frac{\partial B_j}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}(A_jB_j) \\
&= [\nabla(\mathbf{A}\cdot\mathbf{B})]_i
\end{aligned}
$$

as required. ∎

## The divergence of a gradient: the Laplacian

**Definition.** The operation

$$
\begin{aligned}
\operatorname{div}(\nabla\phi) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial z}\,\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{k}}\right) \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
&= \nabla^2\phi
\end{aligned}
$$

This is the *Laplacian* of $\phi$, read as "*del squared* $\phi$". The operator $\nabla^2$ is known as the *Laplacian operator*.

We also define the Laplacian of a vector as

$$\nabla^2\mathbf{A} = \frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2}$$

in Cartesian coordinates, and the equation $\nabla^2\phi = 0$ is known as *Laplace's equation*, which we will see more of in the chapter on partial differential equations.

**Example 1.6.** *If $\phi = x^2 + y^2$, find $\nabla^2\phi$.*

$$\frac{\partial^2\phi}{\partial x^2} = 2 = \frac{\partial^2\phi}{\partial y^2}; \quad \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\implies \nabla^2\phi = 4$$

This is a special case of *Poisson's equation*.

## The curl of a gradient

Consider the operation

$$\begin{aligned}\operatorname{curl}(\nabla\phi) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix} \\ &= \hat{\mathbf{i}}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right) - \hat{\mathbf{j}}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}\right) \\ &= \mathbf{0}\end{aligned}$$

(This result can also be established by using tensor notation.)

**Example 1.7.** Consider $\phi = axy^2 + byz + cx^3z^2$ and show explicitly $\operatorname{curl}\nabla\phi = \mathbf{0}$.

$$\nabla\phi = \hat{\mathbf{i}}(ay^2 + 3cx^2z^2) + \hat{\mathbf{j}}(2axy + bz) + \hat{\mathbf{k}}(by + 2cx^3z)$$

$$\begin{aligned}\implies \operatorname{curl}(\nabla\phi) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ ay^2 + 3cx^2z^2 & 2axy + bz & by + 2cx^3z \end{vmatrix} \\ &= \hat{\mathbf{i}}(b - b) - \hat{\mathbf{j}}(6cx^2z - 6cx^2z) + \hat{\mathbf{k}}(2ay - 2ay) \\ &= \mathbf{0}\end{aligned}$$

as expected.

## The divergence of a curl

This is also always zero, as can be seen from the following argument: Lecture 5

$$\operatorname{div}(\operatorname{curl}\mathbf{A}) = \frac{\partial}{\partial x_i}[\operatorname{curl}\mathbf{A}]_i = \epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_k = 0$$

with the zero following from symmetry argument.

**Example 1.8.** Verify that $\operatorname{div}(\operatorname{curl}\mathbf{A}) = 0$ for the quantity $\mathbf{A} = ye^x\,\hat{\mathbf{i}} + (x^2+z)\,\hat{\mathbf{j}} + y^3\cos(zx)\,\hat{\mathbf{k}}$.

$$\begin{aligned}
\operatorname{curl}\mathbf{A} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ ye^x & x^z + z & y^3\cos(zx) \end{vmatrix} \\
&= \hat{\mathbf{i}}(3y^2(\cos(zx) - 1) - \hat{\mathbf{j}}(-y^3 z\sin(zx)) + \hat{\mathbf{k}}(2x - e^x)
\end{aligned}$$

$$\begin{aligned}
\implies \operatorname{div}(\operatorname{curl}\mathbf{A}) &= \cancel{-3y^2z\sin(zx)} + \cancel{3y^2z\sin(zx)} + 0 \\
&= 0
\end{aligned}$$

as required.

## The curl of a curl

This is the vector quantity $\operatorname{curl}(\operatorname{curl}\mathbf{A})$. Using tensor notation, and the summation convention we can show that

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}) = \nabla(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A}$$

*Proof.*

$$\begin{aligned}
[\operatorname{curl}(\operatorname{curl}\mathbf{A})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\operatorname{curl}\mathbf{A})_k \\
&= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left[\varepsilon_{klm}\frac{\partial A_m}{\partial x_l}\right] \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}A_m \\
&= \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}A_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}A_i \\
&= \frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}A_j\right] - \frac{\partial^2}{\partial x_j^2}A_i \\
&= \frac{\partial}{\partial x_i}[\operatorname{div}\mathbf{A}] - (\nabla^2\mathbf{A})_i \\
&= [\nabla(\operatorname{div}\mathbf{A})]_i - (\nabla^2\mathbf{A})_i
\end{aligned}$$

as required. ∎

*Exercise:* Calculate $\operatorname{curl}(\operatorname{curl}\mathbf{A})$, $\nabla(\operatorname{div}\mathbf{A})$ and $\nabla^2\mathbf{A}$ for $\mathbf{A} = ye^x\,\hat{\mathbf{i}} + (x^2+z)\,\hat{\mathbf{j}} + y^3\cos(zx)\,\hat{\mathbf{k}}$.

## Scalar and vector fields

**Definition.** If, at each point of a region $V$ of space, a scalar function $\phi$ is defined, we say that $\phi$ is a *scalar field* over the region $V$. Similarly, if a vector function $\mathbf{A}$ is also defined at all points of $V$, then $\mathbf{A}$ is a *vector field* over $V$.

If $\operatorname{curl}\mathbf{A} = \mathbf{0}$, we say that $\mathbf{A}$ is an *irrotational* vector field. If $\operatorname{div}\mathbf{A} = 0$, we say that $\mathbf{A}$ is a *solenoidal* vector field.

An obvious example of a vector field is the position vector $\mathbf{r}$ of a point in space. In three dimensions: $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.

$$\begin{aligned}\operatorname{div}\mathbf{r} &= 3\\ \operatorname{curl}\mathbf{r} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ x & y & z\end{vmatrix} = \mathbf{0}\end{aligned}$$

$$\begin{aligned}|\mathbf{r}| &= r = (x^2+y^2+z^2)^{1/2}\\ \nabla r &= \nabla(x^2+y^2+z^2)^{1/2}\\ &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}\\ &= (x^2+y^2+z^2)^{-1/2}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})\\ &= \mathbf{r}/|\mathbf{r}| = \hat{\mathbf{r}}\end{aligned}$$

**Example 1.9.** *Find* $\nabla^2(1/r) = \nabla\cdot(\nabla(1/r))$. $(r \neq 0)$

First

$$\begin{aligned}\nabla(1/r) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}\\ &= (x^2+y^2+z^2)^{-3/2}(-x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}})\end{aligned}$$

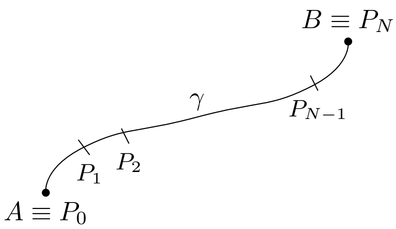
Then

$$\begin{aligned}\nabla\cdot(\nabla(1/r)) = -&\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right.\\ &\left.+\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\right]\\ =& -3(x^2+y^2+z^2)^{-3/2} + (3/2)(2x)(x^2+y^2+z^2)^{-5/2}x\\ +&\quad (3/2)(2y)(x^2+y^2+z^2)^{-5/2}y + (3/2)(2z)(x^2+y^2+z^2)^{-5/2}z\\ =& -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2}\\ =& \ 0\end{aligned}$$

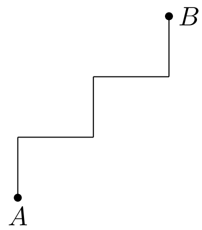
Hence $1/r$ satisfies Laplace's eqn in 3D. In fact $1/(r-r_0)$ is also a solution.

## 1.5 Line integrals

Consider a curve $\gamma$ (not necessarily in the plane, not necessarily smooth) joining the points $A$ and $B$. E.g. Lecture 6




or




**Definition.** Suppose that the curve is divided into $N$ sections: $AP_1, \dots, P_{N-1}B$. Let $AP_1 = \delta s_1$ , $P_1P_2 = \delta s_2, \dots, P_{N-1}B = \delta s_N$. Next, suppose a function $f$ is defined along this curve $\gamma$. We compute the sum

$$f_1 \delta s_1 + f_2 \delta s_2 + \cdots + f_N \delta s_N$$

where $f_n = f(P_N)$.

On increasing $N$ indefinitely, while letting the maximum $\delta s_n \to 0$, the results limit of the sum, if it exists, is called the *line integral of $f$ along $\gamma$* (of first kind), and we write

$$\int_\gamma f \, \mathrm{d}s = \lim_{\substack{N \to \infty \\ \max \delta s_n \to 0}} \sum_{n=1}^{N} f_n \delta s_n$$

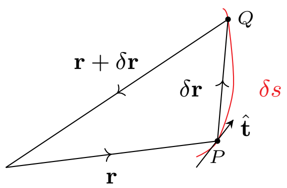
The function $f$ may be a scalar or a vector.

### Line element

**Definition.** Let $\delta s$ represent the arc $PQ$ and suppose that the vector $\vec{PQ} = \delta \mathrm{r}$. We define the *tangent vector*

$$\hat{\mathbf{t}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s}$$

and the *line element* as $\mathrm{d}\mathbf{r} = \hat{\mathbf{r}} \mathrm{d}s$.




$|\delta \mathbf{r}| \to \delta s$
as $\delta s \to 0$

Note that $\hat{\mathbf{t}}$ has length unity because $|\delta \mathbf{r}| \to \delta s$ as $\delta s \to 0$. We can then define

**Definition** (Line integral of 2nd kind)**.**

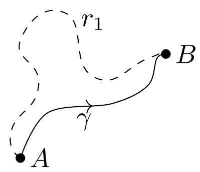
$$\int_\gamma \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_\gamma \underbrace{(\mathbf{A}\cdot\hat{\mathbf{t}})}_{\text{scalar}}\,\mathrm{d}s$$

## Conservative forces

Consider the special case where we have a vector $\mathbf{F}$ of the form $\mathbf{F} = \nabla\phi$. Consider the integral (with $\gamma$ defined as above):

$$\begin{aligned}\int_\gamma \mathbf{F}\cdot \mathrm{d}r &= \int_\gamma (\nabla\phi\cdot\hat{\mathbf{t}})\,\mathrm{d}s \\ &= \int_\gamma \frac{\partial\phi}{\partial s}\,\mathrm{d}s \\ &= [\phi]_A^B \\ &= \phi(B)-\phi(A)\end{aligned}$$

We note that the result is *independent of the path* $\gamma$ joining $A$ to $B$. In particular, if $\gamma$ is a closed curve (i.e. $\mathbf{B}=\mathbf{A}$), then we have $\int_\gamma \mathbf{F}\cdot\mathrm{d}r = \mathbf{0}$.

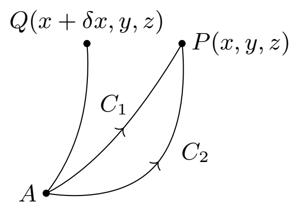



**Definition** (Conservative Field, Potential)**.** If a vector field $\mathbf{F}$ has the property that $\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ for *any* closed curve $\gamma$, we say that $\mathbf{F}$ is a *conservative field.* Conversely, if $\mathbf{F}$ is conservative we can always find a scalar function $\phi$ such that $\mathbf{F}=\nabla\phi$. The function $\phi$ is called the *potential* of the field $\mathbf{F}$.

N.B. $\oint$ means we go all the way round the closed curve.

**Proposition 1.10.** *If* $\mathbf{F}$ *is conservative,* $\exists\,\phi$ *such that* $F=\nabla\phi$*.*

*Proof.* Let $\mathbf{F} = F_1\,\hat{\mathbf{i}}+F_2\,\hat{\mathbf{j}}+F_3\,\hat{\mathbf{k}}$. Since we know that $\mathbf{F}$ is conservative, it must be the case that $\int_A^P \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path from $A$ to $P$. Suppoes $C_1$ and $C_2$ are any two curves drawn from $A$ to $P$. Form the closed curve $\Gamma$ formed of travelling from $A$ to $P$ along $C_1$ and then back along $C_2$ from $P$ to $A$.



Since $\mathbf{F}$ is conservative, we have

$$0 = \int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

and hence

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Suppose that the point $A$ is fixed. Then

$$\begin{aligned}\int_A^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= G(P), \text{ say} \\ &= G(x, y, z)\end{aligned}$$

Let $Q$ be the point $(x + \delta x, y, z)$ and let $P$ be the point $(x, y, z)$. Consider the quantity

$$\begin{aligned}G(x + \delta x, y, z) - G(x, y, z) &\equiv \int_A^Q \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_A^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_P^Q \mathbf{F} \cdot \mathrm{d}\mathbf{r}\end{aligned}$$

But we can choose the path from $P$ to $Q$ so that only $x$ varies, in which case $\mathrm{d}\mathbf{r} = \hat{\mathbf{i}}\mathrm{d}x$. Thus

$$G(x + \delta x, y, z) - G(x, y, z) = \int_x^{x+\delta x} F_1 \,\mathrm{d}x$$

From first principles

$$\begin{aligned}\frac{\partial G}{\partial x} &= \lim_{\delta x \to 0} [G(x + \delta x, y, z) - G(x, y, z)]/\delta x \\ &= \lim_{\delta x \to 0} \left( \int_x^{x+\delta x} F_1 \,\mathrm{d}x \right) \Big/ \delta x \\ &= F_1 \quad \text{(L' Hôptial)}\end{aligned}$$

Similarly we can show that

$$F_2 = \frac{\partial G}{\partial y},\ F_3 = \frac{\partial G}{\partial z}$$

Thus if $\mathbf{F}$ is conservative, then a scalar function ($G$ in this case) can be found such that $F = \nabla G$. So actually $G = \phi$. ■

> **Example 1.11.** *For the vector field* $\mathbf{F} = (3x^2 + yz)\,\hat{\mathbf{i}} + (6y^2 + xz)\,\hat{\mathbf{j}} + (12z^2 + xy)\,\hat{\mathbf{k}}$*, find a scalar function* $\phi(x, y, z)$ *such that* $\mathbf{F} = \nabla\phi$*. Hence calculate* $\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r}$*, where* $A = (0, 0, 0)$ *and* $B = (1, 1, 1)$*.*
>
> [We could show that $\operatorname{curl} \mathbf{F} = 0$]

If $\mathbf{F} = \nabla\phi$, then

$$F_1 = 3x^2 + yz = \frac{\partial\phi}{\partial x} \implies \phi = x^3 + xyz + f_1(y,z)$$

$$F_2 = 6y^2 + zx = \frac{\partial\phi}{\partial y} \implies \phi = 2y^3 + xyz + f_2(x,z)$$

$$F_3 = 12z^2 + xy = \frac{\partial\phi}{\partial z} \implies \phi = 4z^3 + xyz + f_3(x,y)$$

$$\implies \phi = x^2 + 2y^3 + 4z^3 + xyz + C$$

Hence

$$\begin{aligned}\int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{r} = [\phi]_A^B &= \phi(1,1,1) - \phi(0,0,0)\\ &= 1+2+4+1+C-C\\ &= 8\end{aligned}$$

## Circulation

**Definition** (Circulation)**.** When $\gamma$ is a closed curve we often use the notation Lecture 7

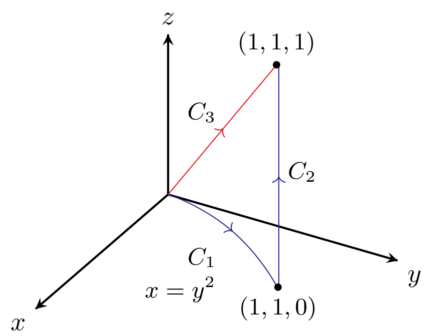
$$\oint_\gamma \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

This integral is called the *circulation* of $\mathbf{F}$ around the closed curve $\gamma$.

**Example 1.12.** *Evaluate* $\int_\gamma \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ with $\mathbf{F} = yz\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + zx\,\hat{\mathbf{k}}$, *when* $\gamma$ *joins* $(0,0,0)$ *to* $(1,1,1)$ *along:*

(i) $C_1 + C_2$ *with the curve* $x = y^2$, $z = 0$ *from* $(0,0,0)$ *to* $(1,1,0)$ *and* $C_2$ *is the straight line joining* $(1,1,0)$ *to* $(1,1,1)$

(ii) $C_3$ *is the straight line joining* $(0,0,0)$ *to* $(1,1,1)$.



(i) On $C_1$: $x = y^2 \implies \mathrm{d}x = 2y\,\mathrm{d}y$. $z = 0 \implies \mathrm{d}z = 0$.

$$\begin{aligned}\mathrm{d}\mathbf{r} &= \mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} + \mathrm{d}z\,\hat{\mathbf{k}} \\ &= 2y\,\mathrm{d}y\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} \text{ along } C_1\end{aligned}$$

Also along $C_1$ :

$$\begin{aligned}\mathbf{F} &= 0\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \\ \implies&\ \mathbf{F}\cdot\mathrm{d}\mathbf{r} = y^3\,\mathrm{d}y \\ \implies&\ \int_{C_1}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 y^2\mathrm{d}y = 1/4\end{aligned}$$

On $C_2$: $x = y = 1 \implies \mathrm{d}x = \mathrm{d}y = 0$. Then $z$ goes from 0 to 1, so $\mathrm{d}\mathbf{r} = \mathrm{d}z\,\hat{\mathbf{k}}$ and $\mathbf{F} = z\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ on $C_2 \implies \mathbf{F}\cdot\mathrm{d}\mathbf{r} = z\,\mathrm{d}z \implies \int_{C_2}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 z\,\mathrm{d}z = 1/2$.

Therefore

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

(ii) Along $C_3 : x = y = z = t$ $(0 \le t \le 1)$. So $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = \mathrm{d}t$ on $C_3 \implies \mathrm{d}\mathbf{r} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\mathrm{d}t$ and $\mathbf{F} = t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$.

$$\begin{aligned}\implies&\ \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 3t^2\,\mathrm{d}t \\ \implies&\ \int_{C_3}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 3t^2\,\mathrm{d}t = [t^3]_0^1 = 1\end{aligned}$$

The answers to (i) and (ii) are different because $\mathbf{F}$ is not conservative i.e. it is not possible to find a $\phi$ such that $\mathbf{F} = \nabla\phi$.

## 1.6 Surface Integrals

**Definition** (Surface Integral)**.** To define a surface integral of $f = f(P)$ over a surface $S$, we divide $S$ into elements of area $\delta S_1, \delta S_2, \ldots, \delta S_N$. Let $f_1, f_2, \ldots, f_S$ be the values of $f$ at typical points $P_1, P_2, \ldots, P_N$ of $\delta S_1, \delta S_2, \ldots, \delta S_N$ respectively. We calculate the quantity

$$\sum_{n=1}^{N} f_n\,\delta S_n$$

We now let $N \to \infty$, $\max\delta S_N \to 0$. The resulting limit, if it exists, is called the *surface integral* of $f$ over $S$, and we write it as

$$\int_S f\,\mathrm{d}S = \lim_{\substack{N\to\infty \\ \max(\delta S_n)\to 0}} \left[\sum_{n=1}^{N} f_n\,\delta S_n\right]$$

## Types of surfaces

*Closed surface*: This divides three-dimensional space into two non-connected regions - an interior region and exterior region. Lecture 8

*Simple closed surface:* This is a closed surface which does not intersect itself.

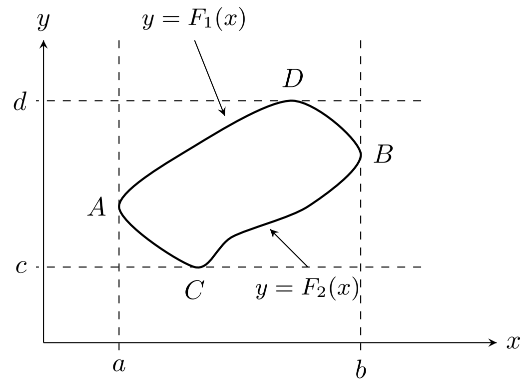
*Convex surface:* This is a surface which is crossed by a straight line at most twice.

*Open surface:* This is does not divide into two non-connected regions - it has a rim which can be represented by a closed curve. (A closed surface can be thought of as the sum of two open surfaces.)

## Evaluation of Surface Integrals

An *areal element* $\mathrm{d}S$ is an 'infinitesimally small' element of area of a surface. Even for closed surfaces it can be thought of as approximately plane. The vector areal element $\mathrm{d}\mathbf{S}$ is the vector $\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the unit vector normal to $\mathrm{d}S$. For plane surfaces $\mathrm{d}S$ can be expressed in Cartesian coordinates $x, y)$ since we may choose the surface to lie in the plane $z = 0$.

Thus we can write $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$. (See figure below).




Let the rectangle $x = a, b$ and $y = c, d$ circumscribe $S$. We will assume for simplicity that $S$ is convex. (If it isnt then we split $S$ up into convex sub-regions). Let the equation of the boundary of $S$ be denoted by

$$y = \begin{cases} F_1(x) & \text{upper half } ADB \\ F_2(x) & \text{lower half } ACB \end{cases}$$

(N.B. we need to ensure these are single-valued functions, which they will be if $S$ is convex.) Then

$$\begin{aligned} S = \int_S \mathrm{d}S &= \int_{x=a}^{x=b} \left[ \int_{y=F_2(x)}^{y=F_1(x)} \mathrm{d}y \right] \mathrm{d}x \\ &= \int_a^b (F_1(x) - F_2(x))\,\mathrm{d}x \end{aligned}$$

If $f(x,y)$ is any function of position:

$$\int_S f \,\mathrm{d}S = \int_{x=a}^{x=b} \left[ \int_{y=F_2(x)}^{y=F_1(x)} f(x,y)\,\mathrm{d}y \right] \mathrm{d}x$$

In some situations it may be more convenient to do the $x$integration first. If we want to do this we need to write the boundaries in terms of functions of $y$ instead of $x$. In this case let the boundary be described by

$$x = \begin{cases} G_1(y) & \text{right half } CBD \\ G_2(y) & \text{left half } CAD \end{cases}$$

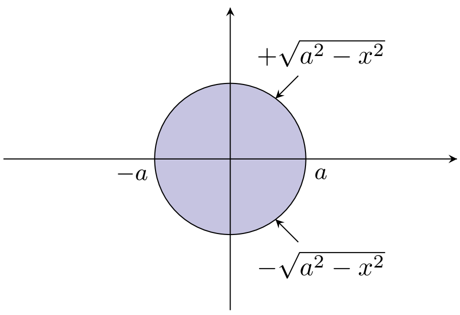
Then

$$\begin{aligned} S = \int_S \mathrm{d}S &= \int_{y=c}^{d} \left[ \int_{x=G_2(y)}^{x=G_1(y)} \mathrm{d}x \right] \mathrm{d}y \\ &= \int_c^d (G_1(y) - G_2(y))\,\mathrm{d}y \end{aligned}$$

and

$$\int_S f \,\mathrm{d}S = \int_{y=c}^{y=d} \left[ \int_{x=G_2(y)}^{G_1(y)} f(x,y)\,\mathrm{d}x \right] \mathrm{d}y$$

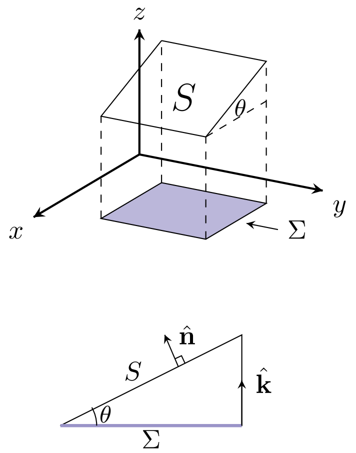
**Example 1.13.** Find the area of the circle $x^2 + y^2 = a^2$.

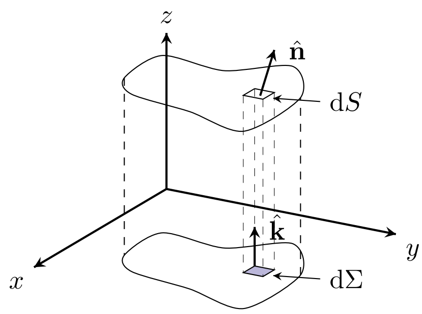



$$\begin{aligned} A &= \int_{x=-a}^{a} \left[ \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} \mathrm{d}y \right] \mathrm{d}x \\ &= 2\int_{-a}^{a} \sqrt{a^2 - x^2}\,\mathrm{d}x \\ &= 2a^2 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \,\mathrm{d}u \\ &= \cdots = \pi a^2 \end{aligned}$$

## Projection of an area onto a plane

Consider first a plane area $S$ (left hand diagram below). Suppose $\Sigma$ is the projected area in the $x - y$ plane. Then, $\Sigma = S\cos\theta$, where $\cos\theta = |\,\hat{\mathbf{n}}\cdot\mathbf{k}|$. Now consider a curved surface. (Right hand diagram below).




If we consider an areal element d$S$ then this will be effectively plane, and so

$$\mathrm{d}S = \frac{\mathrm{d}\Sigma}{|\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}$$

**Theorem 1.14: Projection**

Let $P$ denote a general point of a surface $S$ which at no point is orthogonal to the direction $\hat{\mathbf{k}}$. Then

$$\int_S f(P)\,\mathrm{d}S = \int_\Sigma f(P)\frac{\mathrm{d}x\,\mathrm{d}y}{|\,\hat{\mathbf{n}}\cdot\mathbf{k}|}$$

where $\Sigma$ is the projection of $S$ onto the plane $z = 0$, $\hat{\mathbf{n}}$ normal to $S$.

*Proof.*

$$\begin{aligned}\int_S f(P)\,\mathrm{d}S + &\lim_{\substack{N\to\infty\\ \max(\delta S_r)\to 0}}\sum_{r=1}^{N} f(P_r)\,\delta S_r\\ &= \lim_{\substack{N\to\infty\\ \max(\delta S_r)\to 0}}\sum_{r=1}^{N} f(P_r)\left[\frac{\delta\Sigma_r}{|\,\hat{\mathbf{n}}_r\cdot\hat{\mathbf{k}}|}+\varepsilon_r\right]\end{aligned}$$

where $\varepsilon_r \to 0$ as $\delta S_r \to 0$. (Here $\hat{\mathbf{n}}_r$ is the unit vector normal to $S$ at $PP_r$ and $\delta\Sigma_R$ is the projection of $\delta S_r$ onto the plane $z = 0$. It therefore follows

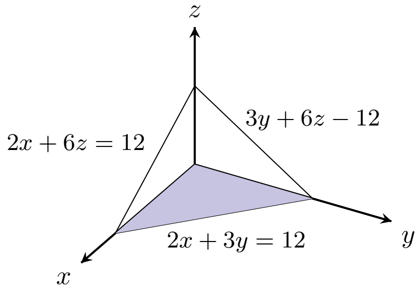
$$\begin{aligned}\int_S f(P)\,\mathrm{d}S &= \int_\Sigma f(x, y, \phi(x, y))\frac{\mathrm{d}x\,\mathrm{d}y}{|\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}\\ &= \int_\Sigma f(p)\frac{d\Sigma}{|\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}\end{aligned}$$

as required. Note that $f(P)$ is evaluated at $P(x,y,z)$ on $S$ in **both integrals** as above. Alternatively, we may choose to project the surface onto $x=0$ or $y=0$ to give:

$$\int_S f(P)\,\mathrm{d}S = \int_{\Sigma_x} f(P)\frac{\mathrm{d}y\,\mathrm{d}z}{|\,\hat{\mathbf{n}}\cdot\hat{\mathbf{i}}|} = \int_{\Sigma_y} f(P)\frac{\mathrm{d}x\,\mathrm{d}z}{|\,\hat{\mathbf{n}}\cdot\hat{\mathbf{j}}|}$$

where $\Sigma_r$ is the projection of $S$ onto $x=0$ and $\Sigma_y$ is the projection of $S$ onto $y=0$. ■

**Example 1.15.** Evaluate $\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the unit outward normal to $S$, $\mathbf{F} = 18z\,\hat{\mathbf{i}} - 12\,\hat{\mathbf{j}} + 3y\,\hat{\mathbf{k}}$ and $S$ is the part of the plane $2x+3y+6z=12$ in the first octant $(x,y,z\geq 0)$. Lecture 9




Normal to the plane $\nabla(2x+3y+6z) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$

$$\Longrightarrow \quad \hat{\mathbf{n}} = \frac{2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}}{\sqrt{4+9+36}} = \frac{1}{7}(2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}})$$

(take the +ve square root to ensure outward normal). Then

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = 6/7 \quad \& \quad \mathbf{F}\cdot\hat{\mathbf{n}} = \frac{1}{7}(36z - 36 + 18y)$$

Project on $z=0$:

*Warning.* Don't put $z=0$; $6z = 12 - 2x - 3y$.

$$\begin{aligned}
\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_\Sigma \frac{1}{7}(36z - 36 + 18y)\frac{\mathrm{d}x\,\mathrm{d}y}{6/7} \\
&= \int_\Sigma \frac{1}{6}(36 - 12x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{x=0}^{6}\left[\int_{y=0}^{y=(12-2x)/3}(6-2x)\,\mathrm{d}y\right]\mathrm{d}x \\
&= \int_0^6 (6-2x)(12-2x)/3\,\mathrm{d}x \\
&= \cdots = 24
\end{aligned}$$

*Exercise:* Try projecting onto $x=0$ or $y=0$

## 1.7 Volume Integrals

**Definition** (Volume Integral)**.** Consider a volume $\tau$ and split it up into $N$ subregions $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$. Let $P_1, P_2, \ldots, P_N$ be typical points of $\delta\tau_1, \ldots, \delta r_N$. Consider the sum

$$\sum_{i=1}^{N} f(P_i)\delta r_i$$

now let $N \to \infty$, $\max \delta r_i \to 0$. If this sum tends to a limit we call it the volume integral of $f$ over $r$ and write this as

$$\int_r f \, \mathrm{d}r$$

The function $f$ may be a vector or a scalar.

**Lemma 1.16.** *If $f$ is a continuous function and $\int_{\tau^*} f \, \mathrm{d}s = \mathbf{0}$ for all subregions $\tau^*$ of $r_i$, then $f = 0$ at every point of $r$.*

*Proof.* Suppose $f > 0$ at $P$ (in $r$). Then it follows that $f > 0$ within a sufficiently small sphere $\tau_\varepsilon$ with centre $P$ and radius $\varepsilon$. Clearly then,

$$\int_{\tau_\varepsilon} f \, \mathrm{d}\tau > 0$$

But $\tau_\varepsilon$ is a subregion of $\tau$, and so we must have

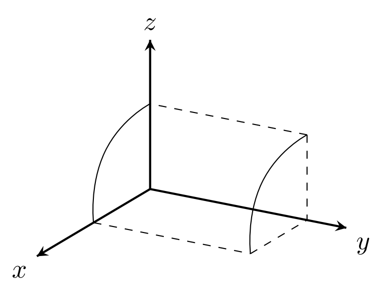
$$\int_{\tau_\varepsilon} f \, \mathrm{d}\tau = 0$$

We therefore have a contradiction and so $f$ cannot be $> 0$ at $P$. Similarly it is not possible for $f$ to be negative at $P$. Thus we conclude that $f$ must be zero at $P$. Since $P$ is an arbitrary point of $\tau$ we conclude that $f = 0$ at all points of $\tau$. ∎

**Definition** (Volume Element)**.**

$$\mathrm{d}\tau = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

**Example 1.17.** Evaluate $\int_\tau (2x + y) \, \mathrm{d}\tau$ when $\tau$ is the volume enclosed by the parabolic cylinder $z = 4 - x^2$ and the plane $x = y = z = 0$ and $y = 2$.



$$
\begin{aligned}
I &= \int_{x=0}^{x=2}\int_{y=0}^{y=2}\int_{z=0}^{z=4-x^2} (2x+y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2\int_0^2 (2x+y)(4-x^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2\int_0^2 (8x-2x^3+4y-x^2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[8xy-2x^3y+2y^2-\frac{x^2y^2}{2}\right]_{y=0}^{y=2}\mathrm{d}x \\
&= \cdots = 80/3
\end{aligned}
$$

## 1.8 Green's, Stoke's and the Divergence Theorem

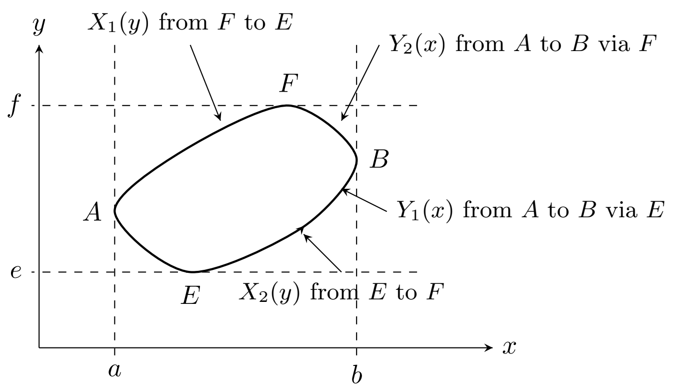
**Theorem 1.18: Green (in the plane)**

Suppose $R$ is a closed plane region bounded by a simple plane closed convex curve in the $x-y$ plane. Let $L, M$ be continuous functions of $x, y$ having continuous derivatives throughout $R$. Then

$$\oint_C (L\mathrm{d}x + M\mathrm{d}y) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathrm{d}x\ \mathrm{d}y$$

where $C$ is the boundary of $R$ described in the counter-clockwise (positive) sense.

*Proof.* We draw a rectangle formed by the tangent lines $x = a, b$ and $y = e, f$. This rectangle circumscribes C. Let $x = X_1(y)$, $x = X_2(y)$ be the equations of $EAF$ and $EBF$ respectively.



We then can write

$$\begin{aligned}\int_R \frac{\partial M}{\partial x}\ \mathrm{d}x\ \mathrm{d}y &= \int_e^f \left[\int_{X_1(y)}^{X_2(y)} \frac{\partial M}{\partial x}\ \mathrm{d}x\right] \mathrm{d}y \\ &= \int_e^f M(X_2(y), y) - M(X_1(y), y)\ \mathrm{d}y \\ &= \int_e^f M(X_2(y), y)\ \mathrm{d}y + \int_f^e M(X_1(y), y)\ \mathrm{d}y \\ &= \oint_C M\,\mathrm{d}y\end{aligned}$$

Now, let the equations of $AEB$ and $AFB$ by $y = Y_1(x)$, $y = Y_2(x)$. Then

$$\begin{aligned}\int_R \frac{\partial L}{\partial y}\ \mathrm{d}x\ \mathrm{d}y &= \int_a^b \left[\int_{Y_1(x)}^{Y_2(x)} \frac{\partial L}{\partial y}\ \mathrm{d}y\right] \mathrm{d}x \\ &= \int_a^b L(x, Y_2(x)) - L(x, Y_1(x))\ \mathrm{d}x \\ &= -\int_a^b L(x, Y_1(x))\ \mathrm{d}x - \int_b^a L(x, Y_2(x))\ \mathrm{d}x \\ &= -\oint_C L\ \mathrm{d}x\end{aligned}$$

■

## Vector forms of Green's Theorem

(i) [2D Stokes' Theorem] Let $\mathbf{F} = L\,\hat{\mathbf{i}} + M\,\hat{\mathbf{j}}$, and $\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}}$. Then Lecture 10

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\hat{\mathbf{k}}$$

Over the region $R$ we can write $\mathrm{d}x\ \mathrm{d}y = \mathrm{d}S$ and $\mathrm{d}\mathbf{S} = \hat{\mathbf{k}}\ \mathrm{d}S$. Thus using Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_R \hat{\mathbf{k}}\cdot\operatorname{curl}\mathbf{F}\ \mathrm{d}S \\ &= \int_R \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S}\end{aligned}$$

This result can be generalised to three dimensions (see Stokes theorem later).

(ii) [Divergence theorem in 2D]. This time let $\mathbf{F} = M\,\hat{\mathbf{i}} - L\,\hat{\mathbf{j}}$ Then

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

and so Green's theorem can be rewritten as

$$\int_R \operatorname{div}\mathbf{F}\ \mathrm{d}x\ \mathrm{d}y = \oint_C F_1\ \mathrm{d}y - F_2\ \mathrm{d}x$$

Now it can be shown (exercise) that

$$\hat{\mathbf{n}}\ \mathrm{d}s = (\mathrm{d}y\,\hat{\mathbf{i}} - \mathrm{d}x\,\hat{\mathbf{j}})$$

here s is arclength along $C$, and $\hat{\mathbf{n}}$ is the unit normal to $C$. Therefore we can rewrite Greens theorem as

$$\int_R \mathrm{div}\mathbf{F}\ \mathrm{d}x\ \mathrm{d}y = \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\ \mathrm{d}s$$

This result also turns out to be true in three dimensions, where it is known as the Divergence Theorem.

**Example 1.19.** *Show that the area enclosed by a simple closed curve with boundary $C$ can be expressed as*

$$\frac{1}{2}\oint_C x\ \mathrm{d}y - y\ \mathrm{d}x$$

*Use this result to calculate the area of an ellipse.*

Green's Theorem:

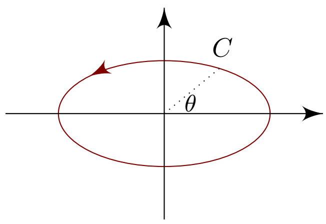
$$\oint_C L\ \mathrm{d}x + M\ \mathrm{d}y = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathrm{d}x\ \mathrm{d}y$$

Let $L = -y$, $M = x$, then

$$\int_C -y\ \mathrm{d}{+x}\ \mathrm{d}y = \int_R (1+1)\ \mathrm{d}x\ \mathrm{d}y = 2\times\ \text{area of } R$$

$$\implies \text{area} = \frac{1}{2}\oint_C x\ \mathrm{d}y - y\ \mathrm{d}x$$

We travel around $C$ to keep finite region to the left:



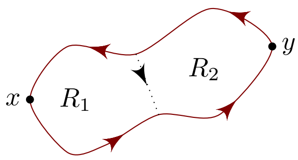

On C: $x = a\cos\theta$, $y = b\sin\theta$ $(0 \le \theta \le 2\pi)$. Then

$$\begin{aligned} x\ \mathrm{d}y - y\ \mathrm{d}x &= (a\cos\theta)(b\cos\theta)\,\mathrm{d}\theta - (b\sin\theta)(-a\sin\theta)\,\mathrm{d}\theta \\ &= ab(\cos^2\theta + \sin^2\theta)\,\mathrm{d}\theta \\ &= ab\,\mathrm{d}\theta \end{aligned}$$

$$\implies \oint (x\ \mathrm{d}y - y\ \mathrm{d}x) = \frac{1}{2}\int_0^{2\pi} ab\ \mathrm{d}\theta = \pi ab$$

## Extensions of Green's Theorem in the plane

Greens theorem is true for more complicated geometries than that assumed in the proof given above. e.g. if $C$ is not convex, but has the shape given below




We can join the points $A, A_0$ so as to form 2 (or more) simple convex closed curves $C_1, C_2$ enclosing $R_1, R_2$ where $R_1 + R_2 = R$. Then:

$$\begin{aligned}\oint_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_{R_1} \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} + \int_{R_2} \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} \\ &= \int_R \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S}\end{aligned}$$

Now

$$\begin{aligned}\oint_{C_1} &= \int_{AXA'} + \int_{A'A} \\ \oint_{C_2} &= \int_{A'YA} + \int_{AA'}\end{aligned}$$

since $\int_{A'A} = -\int_{AA'}$ we have

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_R \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S}$$

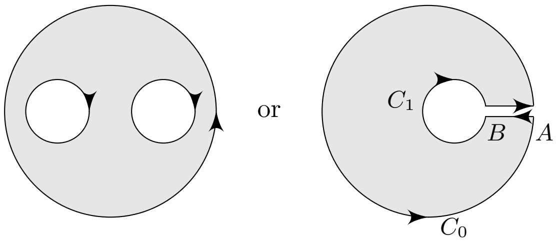
We see therefore that the theorem still holds.

## Green's Theorem in Multiply-Connected Regions

**Definition.** A region $R$ is said to be *simply-connected* if any closed curve drawn in R can be shrunk to a point without leaving R. For example, the interior or exterior of a sphere are both simply-connected, but the exterior of an infinitely long circular cylinder is not simply-connected (left hand picture in figure 11). A region which is not simply-connected is said to be *multiply-connected.*

If $R$ is multiply-connected, Greens theorem is still true provided $C$ is now interpreted as the entire (outer and inner) boundary, with $C$ described so that the region $R$ is always on the left (right hand picture below).

For example if we have a doubly-connected region, we describe the outer boundary $C_0$ in an anti-clockwise fashion and the inner boundary $C_1$ clockwise. We can then join the point $A$ on $C_0$ to the point $B$ on $C_1$ by the line $AB$. This line then divides $R$ in such a way that it is a simply connected region bounded by the closed curve $C_0 + A_B + C_1 + B_A$.




Then by Green's theorem:

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \left( \oint_{C_0} + \int_A^B + \oint_{C_1} + \int_B^A \right) (\mathbf{F} \cdot d\mathbf{r})$$

and therefore it follows that

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

**Definition** (Flux)**.** If $S$ is a surface then the *flux*

$$\int_S \mathbf{A} \cdot d\mathbf{S}$$

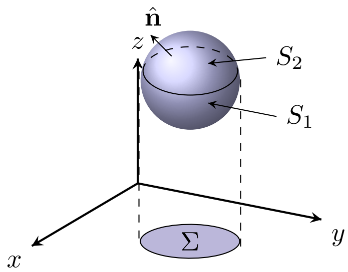
(where $d\mathbf{S} = \hat{\mathbf{k}}\ ds$) If S is a closed surface then, by convention, we always draw the unit normal n out b of S.

**Theorem 1.20: Divergence**

If $\tau$ is the volume enclosed by a closed surface $S$ and $A$ is a vector field with continuous derivatives throughout $\tau$, then:

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_\tau \operatorname{div}\mathbf{A}\ d\tau$$

*Proof.* We will assume that $S$ is convex and that $\tau$ is simply connected, with no interior boundaries.




Let $\mathbf{A} = (A_1, A_2, A_3)$ and $\hat{\mathbf{n}} = (l, m, n)$. (N.B. $|\hat{\mathbf{n}}| \neq n$)

We have to prove that

$$\int_S \underbrace{(lA_1 + mA_2 + nA_3)}_{\mathbf{A}\cdot\hat{\mathbf{n}}} \, \mathrm{d}S = \int_\tau \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z$$

Project $S$ onto the plane $z = 0$. The cylinder with normal cross-section $\Sigma$ and generators parallel to the $z$axis circumscribes $S$ and it touches $S$ along the curve $C$ which divides $S$ into two open surfaces, $S_1$ (upper) and $S_2$ (lower). Both $S_1$ and $S_2$ have projection $\Sigma$ in the plane $z = 0$. Suppose the equations of $S_1$ and $S_2$ are z $= f_1(x, y)$ and $z = f_2(x, y)$ respectively. Then:

$$\begin{aligned} \int_\tau \frac{\partial A_3}{\partial z} \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z &= \int_\tau \frac{\partial A_3}{\partial z} \; \mathrm{d}z \; \mathrm{d}x \; \mathrm{d}y \\ &= \int_\Sigma A_3(x, y, f_2(x, y)) - A_3(x, y, f_1(x, y)) \; \mathrm{d}x \; \mathrm{d}y \end{aligned}$$

Now using the projection theorem Lecture 11

$$\begin{aligned} \int_{S_1} nA_3 \; \mathrm{d}S &= \int_\Sigma nA_3(x, y, f_1(x, y)) \frac{\mathrm{d}x \; \mathrm{d}y}{|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|} \\ &= \int_\Sigma A_3(x, y, f_1(x, y) \; \mathrm{d}x \; \mathrm{d}y \end{aligned}$$

as $|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}| = |n| = n$ since $n > 0$ on $f_1$.

Similarly

$$\begin{aligned} \int_{S_2} nA_3 \; \mathrm{d}S &= \int_\Sigma (nA_3(x, y, f_2(x, y)) \frac{\mathrm{d}x \; \mathrm{d}y}{|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|} \\ &= -\int_\Sigma A_3(x, y, f_2(x, y)) \; \mathrm{d}x \; \mathrm{d}y \end{aligned}$$

as $|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}| = |n| = -n$ since $n < 0$ on $f_2$.

Thus

$$\int_S nA_3 \; \mathrm{d}S = \int_\Sigma [A_3(x, y, f_1(x, y)) - A_3(x, y, f_2(x, y)] \; \mathrm{d}x \; \mathrm{d}y$$

and therefore

$$\int_\tau \frac{\partial A_3}{\partial z} \; \mathrm{d}\tau = \int_S nA_3 \; \mathrm{d}S$$

Similarly, by projecting onto the planes $x = 0$ and $y = 0$:

$$\int_\tau \frac{\partial A_1}{\partial x} \; \mathrm{d}\tau = \int_S lA_1 \; \mathrm{d}S$$

and

$$\int_\tau \frac{\partial A_2}{\partial y} \; \mathrm{d}\tau = \int_S mA_2 \; \mathrm{d}S$$

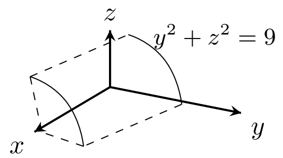
and hence

$$\int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_\tau \operatorname{div} \mathbf{A} \; \mathrm{d}\tau$$

as required. ∎

Note that the surface $S$ need not necessarily be smooth - it could be, for example, a cube or a tetrahedron.

**Example 1.21.** Evaluate $\int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ if $\mathbf{A} = 2x^2y\,\hat{\mathbf{i}} - y^2\,\hat{\mathbf{j}} + 4xz^2\,\hat{\mathbf{k}}$ and $S$ is the surface of the region in the first octant bounded by $y^2+z^2=9$, $x=2$ and $x=y=z=0$.




By the Divergence Theorem

$$\begin{aligned}
\int_V \operatorname{div}\mathbf{A}\;\mathrm{d}V &= \int_V (4xy - 3y + 8xz)\;\mathrm{d}V \\
&= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} ((4xy-3y+8xz))\;\mathrm{d}z\;\mathrm{d}y\;\mathrm{d}x \\
&= \int_0^2\int_0^3 \left[4xyz - 2yz + 8xz^2/2\right]_{z=0}^{z=\sqrt{9-y^2}}\;\mathrm{d}y\;\mathrm{d}x
\end{aligned}$$

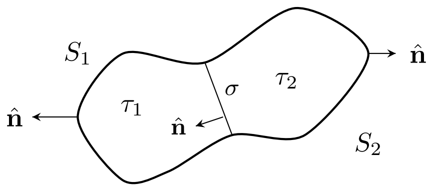
Swapping integration order as integrating $x$ first is easier:

$$\begin{aligned}
&= \int_0^3\int_0^2 4xy(\sqrt{9-y^2}) - 2y\sqrt{9-y^2} + 4x(9-y^2)\;\mathrm{d}x\;\mathrm{d}y \\
&= \int_0^3 \left[2x^2y\sqrt{9-y^2} - 2xy\sqrt{9-y^2} + 2x^2(9-y^2)\right]_{x=0}^{x=2}\;\mathrm{d}y \\
&= \int_0^3 4y\sqrt{9-y^2} + 8(9-y^2)\;\mathrm{d}y \\
&= \cdots = 180
\end{aligned}$$

## Divergence Theorem in more complicated geometries

### (i) Non-convex surfaces

A non-convex surface S can be divided by surface(s) into two (or more) parts $S_1$ and $S_2$ which, together with , form convex surfaces $S_1+\sigma, S_2+\sigma$:




We can then apply the divergence theorem to $S_1+\sigma, S_2+\sigma$ with $\tau_1, \tau_2$ being the respective enclosed volumes, where $\tau_1+\tau_2=\tau$. On adding the results, the surface integrals over $\sigma$ cancel out, and since $S = S_1 + S_2$ we have

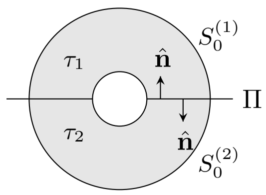
$$\int_S \mathbf{A} \cdot \, \mathrm{d}\mathbf{S} = \int_\tau \operatorname{div} \mathbf{A} \; \mathrm{d}r$$

as before.

**(ii) A region with internal boundaries**

*(a) Simply-connected regions*

For example this could be the space between concentric spheres.



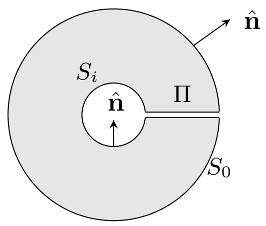

Suppose we have an interior surface $S_i$ and outer surface $S_o$. Draw a plane $\Pi$ that cuts both $S_o$ and $S_i$. This divides $S_o$ into two open surfaces $S_0^{(1)}, S_o^{(2)}$. $S_i$ is similarly divided into $S_i^{(1)}, S_i^{(2)}$. We then apply the divergence theorem to the volume $\tau_1$ which is bounded by the closed surface $S_o^{(1)} + S_i^{(1)} + \Pi$, and we then apply the divergence theorem to the volume $\tau_2$ which is bounded by $S_o^{(2)} + S_i^{(2)} + \Pi$. We add these results together. The contributions over $\Pi$ cancel, leaving the result:

$$\int_{S_0+S_1} \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_{\tau_1+\tau_2} \operatorname{div} \mathbf{A} \; \mathrm{d}\tau = \int_\tau \operatorname{div} \mathbf{A} \; \mathrm{d}\tau$$

with the normal to $S_i$ drawn inwards, i.e. out of $\tau$.

*(b) Multiply-connected regions*

For example this could be the region between two cylinders.




Again let $S_o$ and $S_i$ be the outer and inner surfaces, linked by the plane $\Pi$. Label the two sides of the plane 1 and 2. Consider the surface

$$S_i + \text{ side 1 of } \Pi + S_i + \text{ side 2 of } \Pi$$

This is closed and encloses a simply-connected region $\tau$. We then apply the divergence theorem to $\tau$. The contributions along the two sides of $\Pi$ cancel, giving

$$\int_{S_0+S_1} \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_\tau \operatorname{div} \mathbf{A} \ \mathrm{d}\tau$$

## Green's identities in 3D

Let $\phi$ and $\psi$ be two scalar fields with continuous second derivatives. Consider the quantity Lecture 12

$$\mathbf{A} = \phi \nabla \psi$$

It follows that

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi) \\ \hat{\mathbf{n}} \cdot \mathbf{A} &= \psi \partial \psi / \partial n \end{aligned}$$

Applying the divergence theorem we obtain

$$\int_S \left[ \phi \frac{\partial \psi}{\partial n} \right] \mathrm{d}S = \int_\tau [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] \ \mathrm{d}\tau \tag{1}$$

which is known as *Green's first identity.* Interchanging $\phi$ and $\psi$ we have

$$\int_S \left[ \psi \frac{\partial \phi}{\partial n} \right] \mathrm{d}S = \int_\tau [\psi \nabla^2 \phi + (\nabla \psi) \cdot (\nabla \phi)] \ \mathrm{d}\tau \tag{2}$$

Subtracting (2) from (1) we obtain

$$\int_S \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \mathrm{d}S = \int_\tau (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \ \mathrm{d}\tau$$

which is known as *Green's second identity.* We will use these identities when we analyse the 3D Laplace's equation in the final section of the course.

## Green's identities in 2D

If we use the divergence theorem in 2D derived in the first section of the notes:

$$\int_S \operatorname{div} \mathbf{F} \ \mathrm{d}x \ \mathrm{d}y = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}s$$

then we can calculate down the corresponding Green identities. These are

$$\oint_C \phi \frac{\partial \psi}{\partial n} \ \mathrm{d}s = \int_R \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] \ \mathrm{d}x \ \mathrm{d}y$$

and

$$\oint_C \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \mathrm{d}s = \int_\tau (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \ \mathrm{d}x \ \mathrm{d}y$$

These formulae, (which are the generalisation of integration by parts to two dimensions), will prove useful when considering the 2D version of Laplaces equation later.

**Theorem 1.22: Gauss' Flux**

Let $S$ be a closed surface with outward unit normal $\hat{\mathbf{n}}$, and let $O$ be the origin of the coordinate system. Then

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \begin{cases} 0 & \text{if } O \text{ is exterior to } S \\ 4\pi & \text{if } O \text{ is interior to } S \end{cases}$$

*Proof.* Suppose $O$ is exterior to $S$. Then:

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^2}\,\mathrm{d}S = \int_\tau \mathrm{div}(\mathbf{r}/r^3)\,d\tau$$

using the divergence theorem. But

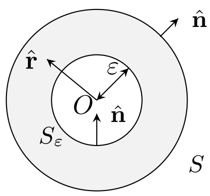
$$\begin{aligned}\mathrm{div}\,(\mathbf{r}/r^3) &= \frac{1}{r^3}\,\mathrm{div}\,\mathbf{r} + \mathbf{r}\cdot\nabla(1/r^3) \\ &= 3/r^3 - \mathbf{r}\cdot(3\mathbf{r}/r^5) = 0\end{aligned}$$

Hence we have that

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^2}\,\mathrm{d}S = 0$$

as required.

Now suppose $O$ is interior to $S$:




We can't apply the divergence theorem directly as before, because $\mathrm{div}(\mathbf{r}/r^3)$ is undefined at the origin ($r = 0$). So we surround $O$ with a small sphere radius $\varepsilon$ with surface $S_\varepsilon$ lying entirely within $S$. We apply the divergence theorem to the region between $S$ and $S_\varepsilon$. Then, as above

$$\int_\tau \mathrm{div}(\mathbf{r}/r^3)\,\mathrm{d}\tau = \int_{S+S_\varepsilon} \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \int_S \frac{\mathbf{r}}{r^3}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_\varepsilon} \frac{\mathbf{r}}{r^3}\cdot\hat{(}-\hat{\mathbf{r}})\,\mathrm{d}S = 0$$

(since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ on $S_\varepsilon$; the normal always points out of the shaded region.)

However on $S_\varepsilon : r = \varepsilon$

$$\int_{S_\varepsilon} \frac{\hat{\mathbf{r}}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \int_{S_\varepsilon} \frac{1}{r^2}\,\mathrm{d}S = \frac{1}{\varepsilon^2}\int_{S_\varepsilon}\mathrm{d}S = \frac{1}{\varepsilon^2}\cdot 4\pi\varepsilon^2 = 4\pi$$

Thus it follows that

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = 4\pi$$

■

**Theorem 1.23: Stokes**

Suppose $S$ is an *open* surface with a simple closed curve $\gamma$ forming its boundary, and let $\mathbf{A}$ be a vector field with continuous partial derivatives. Then

$$\oint_\gamma \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_S \operatorname{curl}\mathbf{A}\cdot \mathrm{d}\mathbf{S}$$

where the direction of the unit normal to $S$ and the sense of $\gamma$ are related by a right-hand rule (i.e. $\hat{\mathbf{n}}$ is in the direction a right-handed screw moves when turned in the direction of $\gamma$).

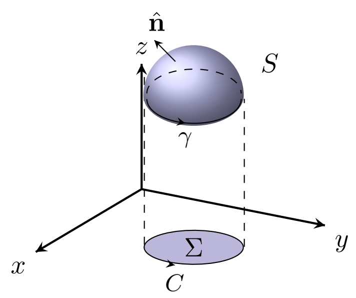
*Proof.* Let $\mathbf{A} = A_1\,\hat{\mathbf{i}} + A_2\,\hat{\mathbf{j}} + A_3\,\hat{\mathbf{k}}$. Consider

$$\operatorname{curl}(A_1\,\hat{\mathbf{i}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ A_1 & 0 & 0 \end{vmatrix} = \hat{\mathbf{j}}\frac{\partial A_1}{\partial z} - \hat{\mathbf{k}}\frac{\partial A_1}{\partial y}$$

Then we have

$$\begin{aligned} \int_S [\operatorname{curl}(A_1\,\hat{\mathbf{i}})]\cdot dS &= \int_S (\,\hat{\mathbf{n}}\cdot \operatorname{curl}(A_1\,\hat{\mathbf{i}})\ \mathrm{d}S \\ &= \int_S \frac{\partial A_1}{\partial z}(\,\hat{\mathbf{j}}\cdot\,\hat{\mathbf{n}}) - \frac{\partial A_1}{\partial y}(\,\hat{\mathbf{k}}\cdot\,\hat{\mathbf{n}})\ \mathrm{d}S \end{aligned}$$

If we now project onto the $x-y$ plane, $S$ becomes $\Sigma$ say, and $\gamma$ becomes $C$ (figure 16):




Let the equation of $S$ be $z = f(x,y)$. Then we have

$$\hat{\mathbf{n}} = \frac{\nabla(z-f(x,y))}{|\nabla(z-f(x,y))|} = \frac{-\partial f/\partial x\,\hat{\mathbf{i}} - \partial f/\partial y\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{((\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1)^{1/2}}$$

Therefore on $S$:

$$\hat{\mathbf{j}}\cdot\,\hat{\mathbf{n}} = -\frac{\partial f}{\partial y}(\,\hat{\mathbf{k}}\cdot\,\hat{\mathbf{n}}) = -\frac{\partial z}{\partial y}(\,\hat{\mathbf{k}}\cdot\,\hat{\mathbf{n}})$$

Thus

$$\int_S [\operatorname{curl}(A_1\,\hat{\mathbf{i}})]\cdot \mathrm{d}S = \int_S \left( -\left.\frac{\partial A_1}{\partial y}\right|_{z,x} - \left.\frac{\partial A_1}{\partial z}\right|_{y,x} - \left.\frac{\partial z}{\partial y}\right|_{x} \right)(\,\hat{\mathbf{k}}\cdot\,\hat{\mathbf{n}})\ \mathrm{d}S$$

Using the transformation rule for partial derivatives

$$= -\int_S \frac{\partial}{\partial y}\bigg|_x A_1(x,y,s)(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})\ \mathrm{d}S$$

Then by the projection theorem:

$$\begin{aligned} &= -\int_\Sigma \frac{\partial}{\partial y} A_1(x,y,f)\ \mathrm{d}x\ \mathrm{d}y \\ &= \oint_C A_1(x,y,f(x,y)) \end{aligned}$$

with the last line following by using Green's theorem. However on $\gamma$ we have $z = f$ and

$$\oint_C A_1(x,y,f)\ \mathrm{d}x = \oint_\gamma A_1(x,y,z)\ \mathrm{d}x$$

We have therefore established that

$$\int_S (\operatorname{curl} A_1\,\hat{\mathbf{i}})\cdot\,\mathrm{d}\mathbf{S} = \oint_\gamma A_1\ \mathrm{d}x$$

In a similar way we can show that

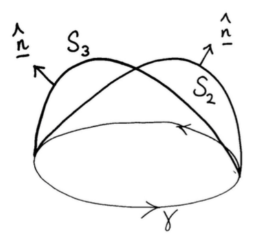
$$\int_S (\operatorname{curl} A_2\,\hat{\mathbf{j}})\cdot\mathrm{d}\mathbf{S} = \oint_\gamma A_2\ \mathrm{d}y$$

and

$$\int_S (\operatorname{curl} A_3\,\hat{\mathbf{k}})\cdot\,\mathrm{d}\mathbf{S} = \oint_\gamma A_3\ \mathrm{d}z$$

and so the theorem is proved by adding all three results together. ■

Note that although $S$ must be open, it is not necessarily smooth. For example it could be in the shape of a box without a lid. The theorem is true for **any** open surface with boundary $\gamma$, since if $S_1$ and $S_2$ are two such surfaces then $S_1 + S_2$ is closed, i.e. Lecture 13




We can then apply the divergence theorem to curl A over the region between $S_1$ and $S_2$, and this gives (since $\mathrm{div}(\mathrm{curl}A) \equiv 0$)

$$\int_{S_1} \operatorname{curl}\mathbf{A}\cdot\mathrm{d}\mathbf{S} - \int_{S_2} \operatorname{curl}\mathbf{A}\cdot\mathrm{d}\mathbf{S} = 0$$

**Theorem 1.24**

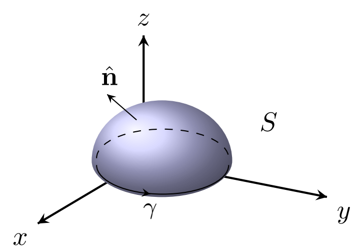
A necessary and sufficient condition that $\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0$ for any closed curve $\gamma$ is that $\operatorname{curl} \mathbf{A} = 0$ throughout the region in which $\gamma$ is drawn.

*Proof.* We already know that if $\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0$ then there exists a potential $\phi$ such that $\mathbf{A} = \nabla\phi$. Therefore we see that $\operatorname{curl} \mathbf{A} = 0$ since the curl of a gradient is always zero.

Conversely, if $\operatorname{curl} \mathbf{A} = 0$ then by Stokes' theorem, we have $\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0$ for any closed curve $\gamma$. ■

**Example 1.25.** *Verify Stokes theorem for the vector field* $\mathbf{A} = (y, z, x)$ *and the surface* $S$ *given by* $z = 1 - x^2 - y^2$ *with* $z \geq 0$.

$$\begin{aligned} \mathbf{A} &= y\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}} \\ \implies \operatorname{curl}\mathbf{A} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} \\ &= -\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} \end{aligned}$$




By the right hand rule, if $\hat{\mathbf{n}}$ points to increasing $z$, we need to move anti-clockwise around $\gamma$ as viewed from above.

**Path integral:**

On $\gamma : z = \mathrm{d}z = 0$ and so $\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} \implies \mathbf{A} \cdot \mathrm{d}\mathbf{r} = y\ \mathrm{d}x + \cancel{z\ \mathrm{d}y}$, so

$$\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{r} = \int y\ \mathrm{d}x$$

on $\gamma$ let $x = \cos\theta$, $y = \sin\theta$ $(0 \leq \theta \leq 2\pi)$, so

$$\begin{aligned} &= -\int_0^{2\pi} \sin^2\theta\ \mathrm{d}\theta \\ &= -\pi \end{aligned}$$

**Surface integral**:

We need

$$\int_S (\operatorname{curl} \mathbf{A}) \cdot \hat{\mathbf{n}} \; \mathrm{d}S$$

where $S$ is the curved surface of a paraboloid.

$$\begin{aligned}\hat{\mathbf{n}} &= \nabla(z - (1 - x^2 - y^2))/|\nabla(\;)| \\ &= (2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + \hat{\mathbf{k}})/\sqrt{(4x^2 + 4y^2 + 1)}\end{aligned}$$

[Choosing $+ve$ square root so that $\hat{\mathbf{n}}$ points towards increasing $z$]

$$\int_S (\operatorname{curl} \mathbf{A}) \cdot \hat{\mathbf{n}} \; \mathrm{d}S = -\int_S \frac{2x + 2y + 1}{\sqrt{(4x^2 + 4y^2 + 1)}} \; \mathrm{d}S$$

Now we use the projection theorem to project onto $z = 0$

$$= -\int_\Sigma \frac{2x + 2y + 1}{\sqrt{(\text{stuff})}} \frac{\mathrm{d}x \; \mathrm{d}y}{|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

Since $|\,\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}| = 1/\sqrt{(\text{stuff})}$, we get

$$= -\int_{x^2+y^2 \le 1} (2x + 2y + 1) \; \mathrm{d}x \; \mathrm{d}y$$

Let $x = r\cos\theta$, $y = r\sin\theta$ for $(0 \le \theta \le 2\pi)$ and $(0 \le r \le 1)$. As $\mathrm{d}x \; \mathrm{d}y = r \; \mathrm{d}r \; \mathrm{d}\theta$ (see later), we have

$$\begin{aligned}&= -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (2r\cos\theta + 2r\sin\theta + 2)r \; \mathrm{d}r \; \mathrm{d}\theta \\ &= -\pi\end{aligned}$$

## 1.9 Curvilinear coordinates

Often it is more convenient, depending on the geometry of the problem under considera- tion, to use coordinates other than Cartesians. An example is cylindrical polar coordinates $(r, \theta, z)$ which are related to Cartesian coordinates by

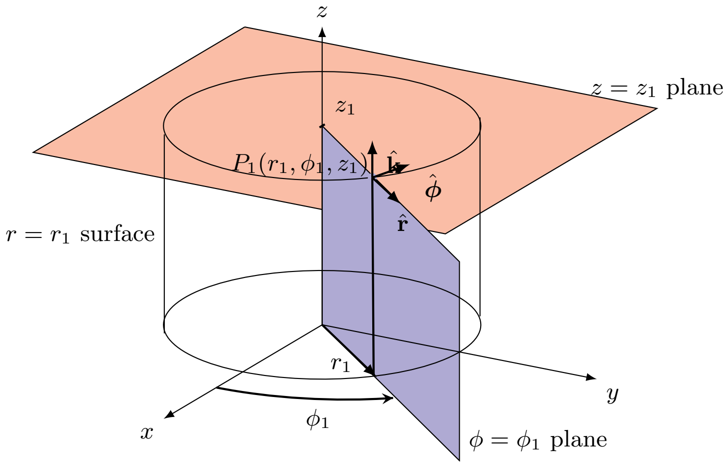
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = (0 \le \theta \le 2\pi,\, 0 \le r < \infty)$$

from which we can deduce that

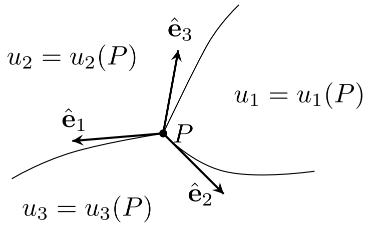
$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

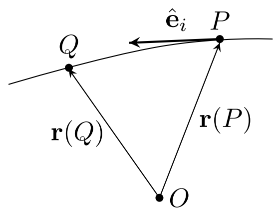
The equation $r =$ constant therefore defines a family of circular cylinders with axes along the z-axis, while the equation $\theta =$ constant defines a family of planes, as does the equation $z =$ constant (picture below).

Cylindrical polar coordinates are an example of *curvilinear coordinates.* The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{k}}$ at any point $P$ are perpendicular to the surfaces $r =$ constant, $\theta =$ constant, $z =$ constant through $P$ in the directions of increasing $r, \theta, z$. Note that the direction of the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ vary from point to point, unlike the corresponding Cartesian unit vectors.




More generally a curvilinear coordinate system is defined in the following way. Suppose that $(x, y, z)$ are expressible as single-valued functions of the variables $(u_1, u_2, u_3)$. Suppose also that $(u_1, u_2, u_3)$ can be expressed as single-valued functions of $(x, y, z)$. Then the equations $u_1 =$ constant, $u_2 =$ constant, $u_3 =$ constant define three families of surfaces, and $(u_1, u_2, u_3)$ is said to be a curvilinear coordinate system.




Through each point $P(x, y, z)$ there passes one member of each family. Let $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ be unit vectors at $P$ in the directions normal to $u_1 = u_1(P)$, $u_2 = u_2(P), u_3 = u_3(P)$ respectively, such that $u_1, u_2, u_3$ increase in the directions $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$. Clearly we must have

$$\hat{\mathbf{a}}_i = \nabla(u_i)/|\nabla u_i|$$

If $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ are mutually orthogonal, the coordinate system is said to be an orthogonal curvilinear coordinate system.

The surfaces $u_2 = u_2(P)$ and $u_3 = u_3(P)$ intersect in a curve, along which only $u_1$ varies. Let $\hat{\mathbf{e}}_1$ be the unit vector tangential to the curve at P. Let $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ be unit vectors tangential to curves along which only $u_2, u_3$ vary. For an orthogonal system we must have $\hat{\mathbf{e}}_i = \hat{\mathbf{a}}_i$ (left diagram above). Let $Q$ be a neighbouring point to P on the curve along which only $u_i$ varies (right diagram above).

We have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u_i} &= \lim_{P\to Q} \frac{(\mathbf{r}(Q)-\mathbf{r}(P))}{\delta u_i} \\ &= \lim_{P\to Q} \frac{(\mathbf{r}(Q)-\mathbf{r}(P))}{PQ} \cdot \lim_{P\to Q}\left(\frac{PQ}{\delta u_i}\right) \\ &= \lim_{Q\to P} \frac{\vec{PQ}}{PQ} \cdot \lim_{P\to Q}\left(\frac{PQ}{\delta u_i}\right) \\ &= \hat{\mathbf{e}}_i \cdot h_i\end{aligned}$$

where we have defined $h_i = |\partial \mathbf{r}/\partial u_i|$. The quantities $h_i$ are often known as the *length scales* for the coordinate system.

**Definition** (Line Element)**.** Since $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$, the *line element* $\mathrm{d}\mathbf{r}$ is given by Lecture 14

$$\begin{aligned}\mathrm{d}\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial \mathbf{r}}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial \mathbf{r}}{\partial u_3}\,\mathrm{d}u_3 \\ &= h_1\,\mathrm{d}u_1\,\hat{\mathbf{e_1}} + h_2\,\mathrm{d}u_2\,\hat{\mathbf{e_2}} + h_3\,\mathrm{d}u_3\,\hat{\mathbf{e_3}}\end{aligned}$$

If the system is orthogonal, then it follows that

$$(\mathrm{d}s)^2 = (\mathrm{d}\mathbf{r})\cdot(\mathrm{d}\mathbf{r}) = h_1^2(\mathrm{d}u_1)^2 + h_2^2(\mathrm{d}u_2)^2 + h_3^2(\mathrm{d}u_3)^2$$

In what follows we will assume we have an orthogonal system so that

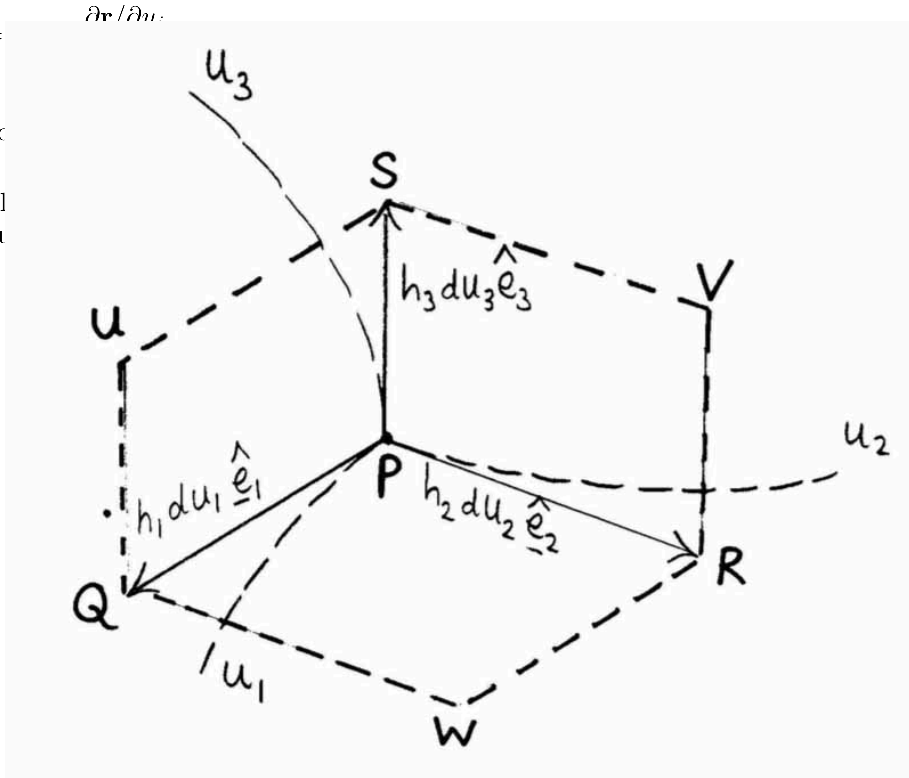
$$\hat{\mathbf{e}}_i = \quad \partial \mathbf{r}/\partial u_i$$

In particular, line

lengths $h_1\,\mathrm{d}u_1$, $h_2$ c

**Definition** (Vol

rectangular (figu



## Line and volume elements in various orthogonal coordinate systems

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**(ii) Cylindrica**
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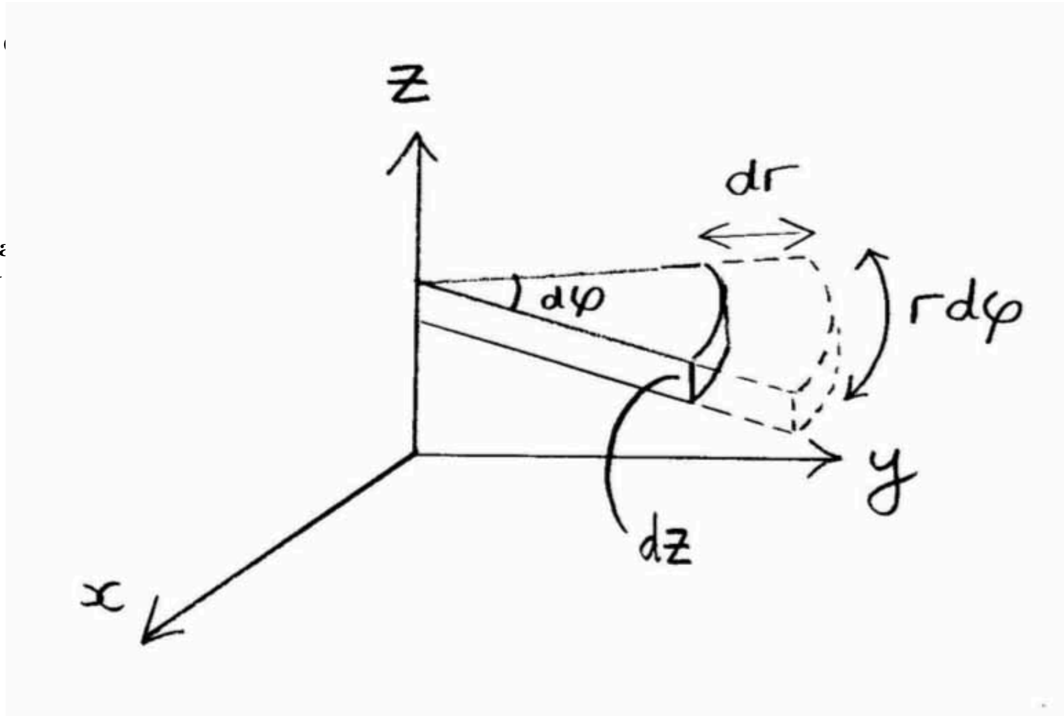



We have that $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$, but we can write

$$\begin{aligned}\mathrm{d}x &= \left(\frac{\partial x}{\partial r}\right)\,\mathrm{d}r + \left(\frac{\partial x}{\partial \phi}\right)\,\mathrm{d}\phi + \left(\frac{\partial x}{\partial z}\right)\,\mathrm{d}z\\ &= (\cos\phi)\,\mathrm{d}r - (r\sin\phi)\,\mathrm{d}\phi\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}y &= \left(\frac{\partial y}{\partial r}\right)\,\mathrm{d}r + \left(\frac{\partial y}{\partial \phi}\right)\,\mathrm{d}\phi + \left(\frac{\partial y}{\partial z}\right)\,\mathrm{d}z\\ &= (\sin\phi)\,\mathrm{d}r + (r\cos\phi)\,\mathrm{d}\phi\end{aligned}$$

Therefore we have

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2\\ &= \cdots = (\mathrm{d}r)^2 + r^2(\mathrm{d}\phi)^2 + (\mathrm{d}z)^2\end{aligned}$$

Thus we see that for this coordinate system, the length scales are

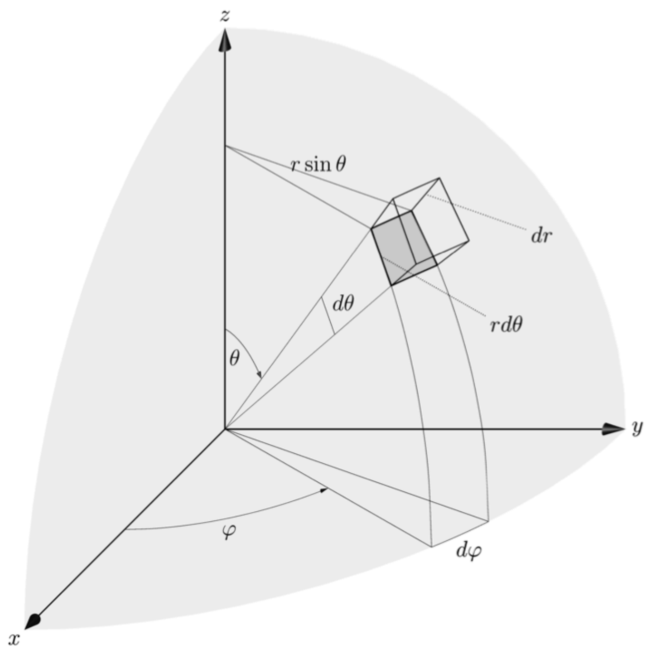
$$h_1 = 1,\; h_2 = r,\; h_3 = 1$$

and the element of volume is

$$d\tau = r\;\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$$

**(iii) Spherical Polar coordinates** $(r, \theta, \phi)$ In this case the relationship between the coordinates is

$$x = r\sin\theta\cos\phi;\; y = r\sin\theta\sin\phi;\; z = r\cos\theta$$




Again, we have that $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$ and we know that

$$\begin{aligned}\mathrm{d}x &= \frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta \\ &= (\sin\theta\cos\phi)\,\mathrm{d}r + (-r\sin\theta\sin\phi)\,\mathrm{d}\phi + r\cos\theta\cos\phi\,\mathrm{d}\theta\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}y &= \frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta \\ &= \sin\theta\sin\phi\,\mathrm{d}r + r\sin\theta\cos\phi\,\mathrm{d}\phi + r\cos\theta\sin\phi\,\mathrm{d}\theta\end{aligned}$$

together with

$$\begin{aligned}\mathrm{d}z &= \frac{\partial z}{\partial r}\,\mathrm{d}r + \frac{\partial z}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial z}{\partial \theta}\,\mathrm{d}\theta \\ &= (\cos\theta)\,\mathrm{d}r - (r\sin\theta)\,\mathrm{d}\theta\end{aligned}$$

Therefore in this case, we have (after some work)

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \\ &= \cdots = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2\end{aligned}$$

Thus the length scales are

$$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$$

and the volume element is

$$\mathrm{d}\tau = r^2\sin\theta\ \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

**Example 1.26.** *Find the volume and surface area of a sphere of radius a, and also find the surface area of a cap of the sphere that subtends on angle $\alpha$ at the centre of the sphere.*

$$d\tau = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

and an element of surface of a sphere of radius $a$ is (by removing $h_1 \, \mathrm{d}u_1 = \mathrm{d}r$):

$$\mathrm{d}S = a^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$\therefore$ total volume is

$$\begin{aligned}\int_\tau \mathrm{d}\tau &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= 2\pi[-\cos\theta]_0^\pi \int_0^a r^2 \, \mathrm{d}r \\ &= 4\pi a^3/3 \; //\end{aligned}$$

Surface area is

$$\begin{aligned}\int_S \mathrm{d}S &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} a^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= 2\pi a^2 [-\cos\theta]_0^\pi \\ &= 4\pi a^2 \; //\end{aligned}$$

Surface area of cap is

$$\begin{aligned}\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} a^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi &= 2\pi a^2 [-\cos\theta]_0^\alpha \\ &= 2\pi a^2 (1 - \cos\alpha)\end{aligned}$$

## Gradient in orthogonal curvilinear coordinates

Let Lecture 15

$$\nabla\Phi = \lambda_1 \, \hat{\mathbf{e}}_1 + \lambda_2 \, \hat{\mathbf{e}}_2 + \lambda_3 \, \hat{\mathbf{e}}_3$$

in a general coordinate system, where $\lambda_1, \lambda_2, \lambda_3$ are to be found. Recall that the element of length is given by

$$\mathrm{d}\mathbf{r} = h_2 \, \mathrm{d}u_1 \, \hat{\mathbf{e}}_1 + h_2 \, \mathrm{d}u_2 \, \hat{\mathbf{e}}_2 + h_3 \, \mathrm{d}u_3 \, \hat{\mathbf{e}}_3$$

Now

$$\begin{aligned}\mathrm{d}\Phi &= \frac{\partial\Phi}{\partial u_1} \mathrm{d}u_1 + \frac{\partial\Phi}{\partial u_2} \mathrm{d}u_2 + \frac{\partial\Phi}{\partial u_3} \mathrm{d}u_3 \\ &= \frac{\partial\Phi}{\partial x} \mathrm{d}x + \frac{\partial\Phi}{\partial u_2} \mathrm{d}y + \frac{\partial\Phi}{\partial z} \mathrm{d}z \\ &= (\nabla\Phi) \cdot \mathrm{d}\mathbf{r}\end{aligned}$$

But, using our expressions for $\nabla\Phi$ and $\mathrm{d}\mathbf{r}$ above:

$$(\nabla\phi) \cdot \mathrm{d}\mathbf{r} = \lambda_1 h_1 \, \mathrm{d}u_1 + \lambda_2 h_2 \, \mathrm{d}u_2 + \lambda_3 h_3 \, \mathrm{d}u_3$$

and so we see that

$$h_i\lambda_i = \frac{\partial\Phi}{\partial u_i}\ (i = 1, 2, 3)$$

Thus we have the result that

$$\nabla\Phi = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

This result now allows us to write down $\nabla$ easily for other coordinate systems.

**(i) Cylindrical polars** $(r, \phi, z)$ Recall that $h_1 = 1$, $h_2 = r$, $h_3 = 1$. Thus

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

**(ii) Spherical Polars** $(r, \theta, \phi)$ We have $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$, and so

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

## Expressions for unit vectors

From the expression for $\nabla$ we have just derived, it is easy to see that

$$\hat{\mathbf{e}}_i = h_i\nabla u_i$$

Alternatively, since the unit vectors are orthogonal, if we know two unit vectors we can find the third from the relation

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = h_2h_3(\nabla u_2 \times \nabla u_3)$$

and similarly for the other components, by permuting in a cyclic fashion.

## Divergence in orthogonal curvilinear coordinates

Suppose we have a vector field

$$\mathbf{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$$

First consider

$$\begin{aligned}\nabla\cdot(A_1\,\hat{\mathbf{e}}_3) &= \nabla\cdot[A_1h_2h_3(\nabla u_2\times\nabla u_3)]\\ &= A_1h_2h_3\nabla\cdot(\nabla u_2\times\nabla u_3) + \nabla(A_1h_2h_3)\cdot\frac{\hat{\mathbf{e}}_1}{h_2h_3}\end{aligned}$$

using the results established just above. Also we know that

$$\nabla\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot\operatorname{curl}\mathbf{B} - \mathbf{B}\cdot\operatorname{curl}\mathbf{C}$$

and so it follows that

$$\nabla\cdot(\nabla u_2\times\nabla u_3) = (\nabla u_3)\cdot\operatorname{curl}(\nabla u_2) - (\nabla u_2)\cdot\operatorname{curl}(\nabla u_3) = 0$$

since the curl of a gradient is always zero. Thus we are left with

$$\nabla\cdot(A_1\,\hat{\mathbf{e}}_1) = \nabla(A_1h_2h_3)\cdot\frac{\hat{\mathbf{e}}_1}{h_2h_3} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(A_1h_2h_3)$$

We can proceed in a similar fashion for the other components, and establish that

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]}$$

It is now easy to write down div in other coordinate systems.

**(i) Cylindrical polars** $(r, \phi, z)$

Recall that $h_1 = 1$, $h_2 = r$, $h_3 = 1$. Thus using the above formula:

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial \phi}(A_2) + \frac{\partial}{\partial z}(rA_3)\right] \\ &= \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}\end{aligned}$$

**(ii) Spherical polars** $(r, \theta, \phi)$

We have $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Hence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta A_1) + \frac{\partial}{\partial \theta}(r\sin\theta A_2) + \frac{\partial}{\partial \phi}(rA_3)\right]$$

## Curl in orthogonal curvilinear coordinates

Again, just consider the curl of the first component of $\mathbf{A}$:

$$\begin{aligned}\nabla \times (A_1\, \hat{\mathbf{e}}_1) &= \nabla \times (A_1 h_1 \nabla u_1) \\ &= A_1 h_2 \nabla \times (\nabla u_1) + \nabla(A_1 h_1) \times \nabla u_1 \\ &= 0 + \nabla(A_1 h_1) \times \nabla u_1 \\ &= \left[\frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_1) + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}(A_1 h_1)\right] \times \frac{\hat{\mathbf{e}}_1}{h_1} \\ &= \frac{\hat{\mathbf{e}}_2}{h_1 h_2}\frac{\partial}{\partial u_3}(h_1 A_1) - \frac{\hat{\mathbf{e}}_3}{h_1 h_2}\frac{\partial}{\partial u_2}(h_1 A_1)\end{aligned}$$

(since $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 = 0,\ \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3,\ \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2$).

We can obviously find curl$(A_2\, \hat{\mathbf{e}}_2)$ and curl$(A_3\, \hat{\mathbf{e}}_3)$ in a similar way. These can be shown to be

$$\begin{aligned}\nabla \times (A_2\, \hat{\mathbf{e}}_2) &= \frac{\hat{\mathbf{e}}_3}{h_2 h_1}\frac{\partial}{\partial u_1}(h_2 A_2) - \frac{\hat{\mathbf{e}}_1}{h_2 h_3}\frac{\partial}{\partial u_3}(h_2 A_2) \\ \nabla \times (A_3\, \hat{\mathbf{e}}_3) &= \frac{\hat{\mathbf{e}}_1}{h_3 h_2}\frac{\partial}{\partial u_2}(h_3 A_3) - \frac{\hat{\mathbf{e}}_2}{h_3 h_1}\frac{\partial}{\partial u_1}(h_3 A_3)\end{aligned}$$

Adding these three contributions together, we find we can write this in the form of a determinant as

$$\boxed{\operatorname{curl} \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\, \hat{\mathbf{e}}_1 & h_2\, \hat{\mathbf{e}}_2 & h_3\, \hat{\mathbf{e}}_3 \\ \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}}$$

in which form it is probably easiest remembered. It's then straightforward to write down curl in various orthogonal coordinate systems.

**(i) Cylindrical polars**

$$\operatorname{curl}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \partial_r & \partial_\phi & \partial_z \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

**(ii) Spherical polars**

$$\operatorname{curl}\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_1 & rA_2 & r\sin\theta A_3 \end{vmatrix}$$

## Alternative definitions for grad, div, curl

Let $\tau$ be a region enclosed by a surface $S$ and let $P$ be a general point of $\tau$. We established earlier that

$$\int_\tau \nabla\phi \; \mathrm{d}\tau = \int_S \hat{\mathbf{n}}\phi \; \mathrm{d}S$$

(Problem Sheet 3) It follows that

$$\int_\tau \hat{\mathbf{i}}\cdot\nabla\phi \; \mathrm{d}\tau = \int_S (\hat{\mathbf{i}}\cdot\hat{\mathbf{n}})\phi \; \mathrm{d}S$$

Now the LHS above can be written as $\tau\overline{\{\hat{\mathbf{i}}\cdot\nabla\phi\}}$, where the bar denotes the mean value of this quantity over $\tau$. Since we are assuming that $\phi$ has continuous derivates throughout $\tau$, we can write

$$\overline{\{\hat{\mathbf{i}}\cdot\nabla\phi\}} = \{\hat{\mathbf{i}}\cdot\nabla\phi\}_Q$$

for some point $Q$ of $\tau$. Thus we have that

$$\{\hat{\mathbf{i}}\cdot\nabla\phi\}_Q = \frac{1}{\tau}\int_S (\hat{\mathbf{i}}\cdot\hat{\mathbf{n}})\phi \; \mathrm{d}S$$

Now let $\tau \to 0$ about $P$. Then $P \to Q$ and we have that at any point $P$ of $\tau$:

$$\hat{\mathbf{i}}\cdot\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\hat{\mathbf{i}}\cdot\hat{\mathbf{n}})\phi \; \mathrm{d}S$$

Similar results can be established for $\hat{\mathbf{j}}\cdot\nabla\phi$ and $\hat{\mathbf{k}}\cdot\nabla\phi$. Taken together, these imply that

$$\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S \hat{\mathbf{n}}\phi \; \mathrm{d}S$$

This can be regarded an an alternative way of definining $\nabla\phi$, rather than defining it as $(\partial\phi/\partial x)\,\hat{\mathbf{i}} + (\partial\phi/\partial u)\,\hat{\mathbf{j}} + (\partial\phi/\partial z)\,\hat{\mathbf{k}}$. We can similarly establish that

$$\operatorname{div}\mathbf{A} = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\hat{\mathbf{n}}\cdot\mathbf{A}) \; \mathrm{d}S$$

$$\operatorname{curl}\mathbf{A} = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\hat{\mathbf{n}}\times\mathbf{A}) \; \mathrm{d}S$$

which are alternati
independent of the
this approach. In
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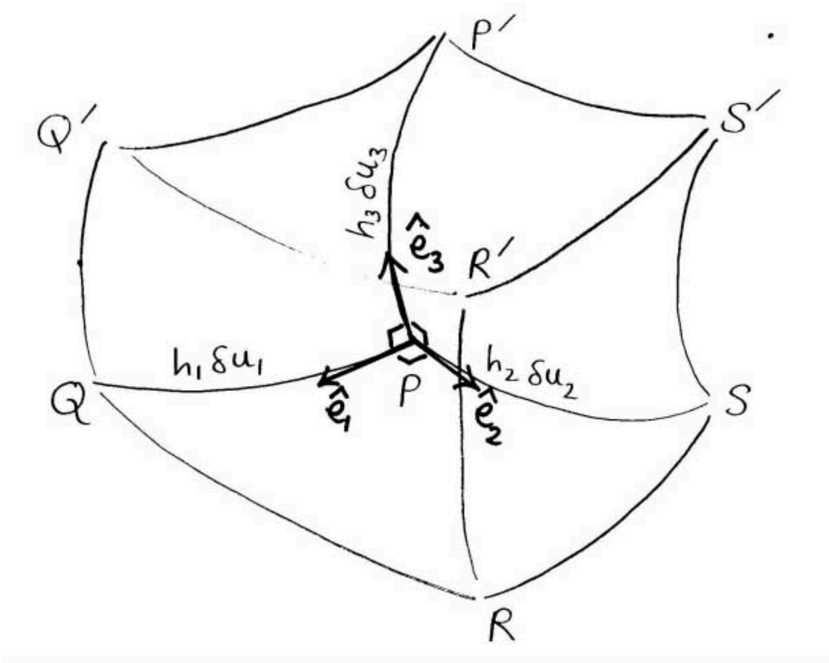
**Equivalence of d**

Lets show that the
curvilinear formula
volume element loc
and unit vectors al




The volume of the element $\delta\tau \approx h_1 h_2 h \, \delta u_1 \delta u_2 \delta u_3$. We start with our definition

$$\operatorname{div} \mathbf{A} = \lim_{\tau \to 0} \frac{1}{\tau} \int_S (\,\hat{\mathbf{n}} \cdot \mathbf{A})\, \mathrm{d}S$$

and aim to compute explicitly the right-hand-side. This involves calculating the contributions to $\int_S$ arising from the six faces of the volume element. If we start with the contribution from the face $PP'S'S$, this is

$$-(A_1 h_2 h_3)_P \delta u_2 \delta u_3 + \text{ higher order terms}$$

The contribution from the face $QQ'R'R$ is

$$(A_1 h_2 h_3)_Q \delta u_3 \delta u_3 + \text{h.o.t} = \left[ (A_1 h_2 h_3) + \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\delta u_1 \right]_P \delta u_2 \delta u_3 + \text{h.o.t}$$

using a Taylor series expansion. Adding together the contributions from these two faces, we get

$$\left[ \frac{\partial}{\partial u_1}(A_1 h_2 h_3) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t.}$$

Similarly the sum of the contributions from the faces $PSRQ$, $P'S'R'Q'$ is

$$\left[ \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t.}$$

while the combined contributions from $PQQ'P'$, $SRR'S'$ is

$$\left[ \frac{\partial}{\partial u_2}(A_2 h_3 h_1) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t.}$$

If we then let $\delta\tau \to 0$, we have that

$$\lim_{\delta\tau\to 0} \frac{1}{\delta\tau}\int_S \hat{\mathbf{n}}\cdot\mathbf{A}\ \mathrm{d}S = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

and so we can see that the integral expression for div $\mathbf{A}$ is consistent with the formula in curvilinear coordinates derived earlier.

### The Laplacian in orthogonal curvilinear coordinates

From the formulae already established for grad and div, we can see that

$$\begin{aligned}\nabla^2\Phi &= \nabla\cdot(\nabla\Phi)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(h_3h_1\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(h_1h_2\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3})\right]\end{aligned}$$

The formula can then be used to calculate the Laplacian for various coordinate systems.

**(i) Cylindrical polars** $(r, \phi, z)$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right]\\ &= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

**(ii) Spherical polars** $(r, \theta, \phi)$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right]\\ &= \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\end{aligned}$$

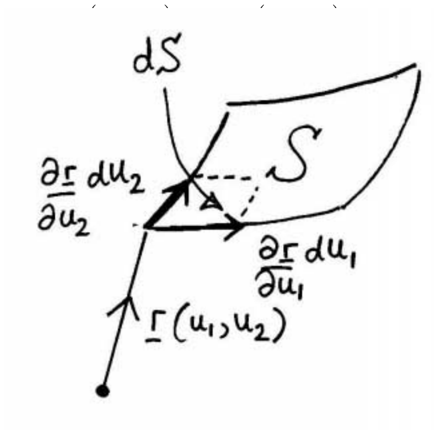
## 1.10 Change of variable in surface integration

Suppose we have a surface $S$ which is parameterised by the quantities $u_1$, $u_2$. We can therefore write that on $S$:

$$x = x(u_1, u$$

[For example, if $S$ is the su … ave $x =$ $\sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z =$ … $\phi$.]

We can consider the surface t… lelograms whose sides are obtained by …



i.e.

$$\begin{aligned} \mathrm{d}S &= \text{ Area of parallelogram with sides } \frac{\partial \mathbf{r}}{\partial u_1}\,\mathrm{d}u_1 \text{ and } \frac{\partial \mathbf{r}}{\partial u_2}\,\mathrm{d}u_2 \\ &= |\mathbf{J}|\ \mathrm{d}u_1\,\mathrm{d}u_2 \end{aligned}$$

**Definition** (Jacobian)**.** The *Jacobian*, $\mathbf{J}$, is given by

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$$

This result is particularly useful when using a substitution in a surface integral, as we can write

$$\int_S f(x,y,z)\ \mathrm{d}S = \int_S F(u_1,u_2)|\mathbf{J}|\ \mathrm{d}u_1\,\mathrm{d}u_2$$

where $F(u_1,u_2) = f(x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$.

If $S$ is a region $R$ in the $x-y$ plane (i.e. $z=0$ on $R$), the result reduces to

$$\int_R f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \int_R F(u_1,u_2)|J|\ \mathrm{d}u_1\,\mathrm{d}u_2$$

where $J$ is now a scalar given by

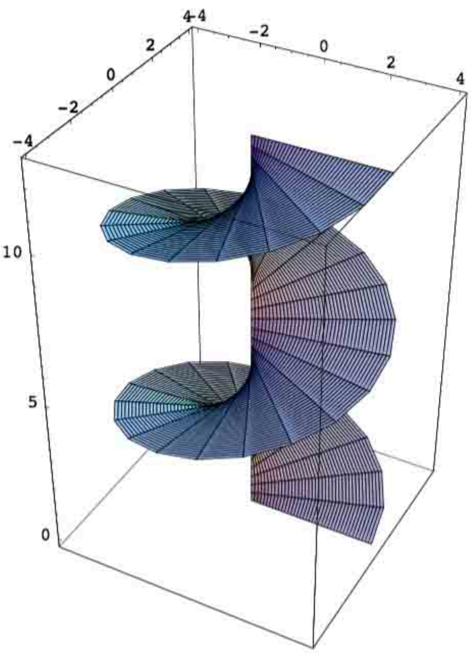
$$J = \begin{vmatrix} \partial x/\partial u_1 & \partial y/\partial u_1 \\ \partial x/\partial u_2 & \partial y/\partial u_2 \end{vmatrix}$$

Note that since $\mathrm{d}x\ \mathrm{d}y = |J|\,\mathrm{d}u_1\,\mathrm{d}u_2$, it follows that $\mathrm{d}u_1\,\mathrm{d}u_2 = (1/|J|)\ \mathrm{d}x\,\mathrm{d}y$, and hence

$$|1/J| = \left|\begin{vmatrix} \partial u_1/\partial x & \partial u_2/\partial x \\ \partial u_1/\partial y & \partial u_2/\partial y \end{vmatrix}\right|$$

which is a useful resul

orthogonal transforma




**Example 1.27.** E 6

the surface of the he

with $0 \leq u \leq 4$ and

We need to find

$$\begin{aligned}
\mathbf{J} &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{vmatrix} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} \\
&= (\sin v)\,\hat{\mathbf{i}} - (\cos v)\,\hat{\mathbf{j}} + \underbrace{(u \cos^2 v + u \sin^2 v)}_{u}\,\hat{\mathbf{k}}
\end{aligned}$$

Therefore

$$|\mathbf{J}| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

Also

$$\sqrt{1 + x^2 + y^2} = \sqrt{1 + u^2 \cos^2 v + u^2 \sin^2 v} = \sqrt{1 + u^2}$$

$$\begin{aligned}
\implies \int_S \sqrt{1 + x^2 + y^2}\ \mathrm{d}S &= \int_S \sqrt{1 + u^2} |\mathbf{J}|\ \mathrm{d}u\,\mathrm{d}v \\
&= \int_{v=0}^{4\pi} \int_{u=0}^{4} (1 + u^2)\ \mathrm{d}u\,\mathrm{d}v \\
&= 4\pi \left[ u + \frac{u^3}{3} \right]_0^4 \\
&= 4\pi \left( 4 + \frac{64}{3} \right) \\
&= \frac{304\pi}{3}
\end{aligned}$$


*"Mom says it's safer."*

# 2 Fourier Series

We will start off with some motivation for our study. Consider a thin metal bar of length 1 which is maintained at zero temperature at the ends $x = 0$ and $x = 1$. At time $t = 0$ the temperature $T(x)$ of the bar is measured - suppose the temperature distribution is $f(x)$. We wish to find the subsequent temperature of the bar, i.e. find $T(x, t)$ for $t > 0$ and $0 < x < 1$. It can be shown that the temperature satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

(We will study this *partial differential equation* in the final part of the course). The constant $\kappa$ is related to the properties of the bar. Lets consider the boundary and initial conditions that need to be applied. Since the heat equation is second order in space and first order in time we would expect to need two space (boundary) conditions and one time (initial) condition. As mentioned above, the temperature is fixed at the ends $x = 0$ and $x = 1$ so this means

$$T(0, t) = T(1, t) = 0$$

The initial condition imposes the initial shape of the temperature distribution, so we have

$$T(x, 0) = f(x) \quad (0 \le x \le 1)$$

with $f(x)$ known. A possible solution to this problem that satisfies both the equation and the two boundary conditions is

$$T_n = b_n \sin(n\pi x)\, e^{-\varkappa n^2 \pi^2 t}$$

where $b_n$ is a constant and $n$ is any integer. (You should check that the form for $T_n$ given here satisfies the heat conduction equation). Each individual $T_n$ is known as a mode. Since the heat equation is linear we can take a linear combination of these solutions to form a 'general' solution

$$T(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-\varkappa n^2 \pi^2 t}$$

where we have assumed that the form of the $b_n$s is such that the infinite series converges. This solution has to be consistent with the initial temperature distribution we imposed, and so applying the initial condition:

$$f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad (0 \le x \le 1) \qquad (1)$$

We conclude that 'any' function $f(x)$ can be written as a superposition of trigonometric oscillations (in this case just sine waves) of appropriate amplitudes (the $b_n$) and frequencies $n$. The right hand side above is known as the Fourier series of $f(x)$ (in this case we actually have what is known as a 'half-range' series - we will come back to this later).

The main problem with (1) is it is by no means obvious how we obtain the so-called Fourier coefficients $b_n$ for a given shape function $f(x)$, i.e. can we make $_b$n the subject of the expression (1)? The answer is yes, and in a surprisingly simple way, but first of all we need to introduce some ideas and definitions

## 2.1 Orthonormal Systems

**Definition.** A sequence of integrable function $\{\phi_i\}_{i=1}^{\infty}$ on an interval $[a, b]$ is called *orthogonal* if

$$\int_a^b \phi_i \phi_j \, \mathrm{d}x = 0 \quad \text{for } i \neq j$$

If in addition

$$\int_a^b \phi_i^2 \, \mathrm{d}x = 1 \quad \text{for all } i$$

the system is said to be *orthonormal.*

**Example 2.1.** The functions

$$\frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}} \cos nx, \ \frac{1}{\sqrt{\pi}} \sin nx, (n = 1, 2, \dots)$$

form an orthonormal system in the interval $[-\pi, \pi]$. To see this we consider the individual cases. First if we consider the orthogonality we see that

$$\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\pi}} \cos nx \, \mathrm{d}x = \frac{1}{\sqrt{2\pi^2}} \frac{1}{\sqrt{\pi}} \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \sin nx \, \mathrm{d}x = \frac{1}{\sqrt{2\pi^2}} \left[\frac{-\cos nx}{n}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \tfrac{1}{\sqrt{\pi}} \cos mx \tfrac{1}{\sqrt{\pi}} \cos nx \, \mathrm{d}x = \tfrac{1}{\pi} \cdot \tfrac{1}{2} \int_{-\pi}^{\pi} \cos[(n+m)x] + \cos[(n-m)x] \, \mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} \tfrac{1}{\sqrt{\pi}} \sin mx \tfrac{1}{\sqrt{\pi}} \cos nx \, \mathrm{d}x = \tfrac{1}{\pi} \cdot \tfrac{1}{2} \int_{-\pi}^{\pi} [\sin[(n+m)x] + \sin[(n-m)x] \, \mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} \tfrac{1}{\sqrt{\pi}} \sin mx \tfrac{1}{\sqrt{\pi}} \sin nx \, \mathrm{d}x = \tfrac{1}{\pi} \cdot \tfrac{1}{2} \int_{-\pi}^{\pi} \cos[(n-m)x] - \cos[(n+m)x] \, \mathrm{d}x = 0$$

Now for the orthonormality:

$$\int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \mathrm{d}x = 1$$

$$\int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{\pi}} \cos nx\right)^2 \mathrm{d}x = \tfrac{1}{\pi} \int_{-\pi}^{\pi} (\tfrac{1}{2} + \tfrac{1}{2} \cos 2nx) \, \mathrm{d}x = \tfrac{1}{2\pi} \left[x + \tfrac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} = 1$$

$$\int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{\pi}} \sin nx\right)^2 \mathrm{d}x = \tfrac{1}{\pi} \int_{-\pi}^{\pi} (\tfrac{1}{2} - \tfrac{1}{2} \cos 2nx) \, \mathrm{d}x = 1$$

Thus the system is shown to be orthonormal. We can write two of the key results as

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \pi \delta_{mn}$$

where $\delta_{mn}$ is the Kronecker delta introduced in Section 1 of the course.

*Exercise:* Show that the functions $(\sqrt{2/\pi})$, $\sin nx$ $(n = 1, 2, \ldots)$ form an orthonormal system on the interval $[0, \pi]$.

## Periodic Functions

**Definition.** A function $f$ is periodic with period $T$ if

$$f(x + T) = f(x) \text{ for all values of } x$$

Obviously $2T, 3T, \ldots,$ will also satisfy this condition, but the period is defined to be the smallest value of $T$.

**Example 2.2.** We have

$$\sin nx = \sin(nx + 2\pi) \equiv \sin\left[n(x + \tfrac{2\pi}{n})\right]$$
$$\cos nx = \cos(nx + 2\pi) \equiv \cos\left[n(x + \tfrac{2\pi}{n})\right]$$

and so $\sin nx$, $\cos nx$ both have period $2\pi/n$. Note that the period gets shorter as the frequency (or wavelength) $n$ increases.

Now consider the finite sum

$$\sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$$

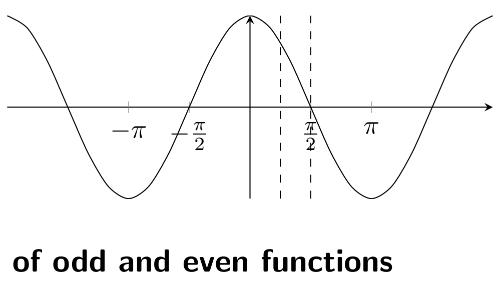
This is a sum of functions with individual periods $2\pi$, $2\pi/2$, $2\pi/3$ etc. The overall period is therefore $2\pi$, i.e. determined by the $n = 1$ mode.

*Exercise:* Find the period of the function $\cos 3x \sin 4x$.

## Odd and even functions

**Definition.** A function $f(x)$ is *even* at $x = a$ if $f(a + x) = f(a - x)$ for all $x$. A function $f(x)$ is *odd* at $x = a$ if $f(a + x) = -f(a - x)$ for all $x$.

**Example 2.3.** Since $\cos(x) = \cos(-x)$ and $\sin(x) = -\sin(-x)$, we have that $\cos x$ is even about $x = 0$ and $\sin x$ is odd about $x = 0$. Note that if we consider $a \neq 0$, these properties change. For example $\cos x$ is neither odd nor even about $x = \pi/4$, and $\cos x$ is odd about $x = \pi/2$. Lecture 17




## Multiplication of odd and even functions

It follows from the definition of odd and even functions, that if $f(x)$ is odd (even) about $x = a$ and $g(x)$ is even (odd) about $x = a$ then the product

$f(x)g(x)$ is odd. Similarly the product of two odd functions is even and the product of two even functions is even, (n.b. odd × odd is even, odd × even is odd, unlike for odd and even numbers!)

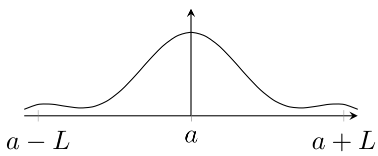
### Integration of odd/even functions

A useful property of odd and even functions that we will make use of is what happens if you integrate them over a range, with the midpoint of the range being the line of symmetry.

If we suppose $f(x)$ is even about $x = a$, then

$$\begin{aligned}
\int_{a-L}^{a+L} f(x)\ \mathrm{d}x &= \int_{a-L}^{a} f(x)\ \mathrm{d}x + \int_{a}^{a+L} f(x)\ \mathrm{d}x \\
&= \int_{-L}^{0} f(u+a)\, du + \int_{a}^{a+L} f(x)\ \mathrm{d}x \\
&= \int_{-L}^{0} f(a-u)\ \mathrm{d}u + \int_{a}^{a+L} f(x)\ \mathrm{d}x \\
&= \int_{a+L}^{a} -f(x)\ \mathrm{d}x + \int_{a}^{a+L} f(x)\ \mathrm{d}x \\
&= 2\int_{a}^{a+L} f(x)\ \mathrm{d}x
\end{aligned}$$

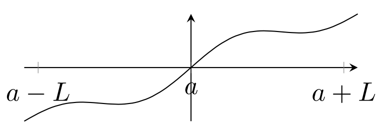
which is obvious if you think of integration as the area under the curve:




Similarly if $g(x)$ is odd about $x = a$ then

$$\int_{a-L}^{a+L} g(x)\ \mathrm{d}x = 0$$

because the areas under the curve between a  L & a, and a & a + L are equal and opposite:




We will use these properties when studying half-range Fourier series later.

## 2.2 Full-range Fourier Series

We begin by considering Fourier series over the range $[-\pi, \pi]$. We will generalise this later. Let $f(x)$ be a periodic function with period $2\pi$. The Fourier series for $f(x)$ is the representation of $f(x)$ as a series in $\sin nx$ and $\cos nx$ of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_n$ $(n \geq 0)$ and $b_n$ $(n \geq 1)$ are constants to be found.

## Finding the Fourier coefficients

We will assume that the series may be integrated term by term, i.e. that integration and summation are interchangeable. First, if we integrate both sides of (2) from $x = -\pi$ to $x = +\pi$ we have

$$\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \pi a_0 + 0 \implies a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x \tag{3}$$

Now if we go back to (2) but this time multiply by $\cos mx$ and integrate from $-\pi$ to $+\pi$, the first term on the RHS is

$$\frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mx\ \mathrm{d}x = 0$$

The second term is

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\ \mathrm{d}x = \sum_{n=1}^{\infty} a_n \pi \delta_{mn} = \pi a_m$$

using the result for the integral calculated earlier in 2.2. Similarly the third term is

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\ \mathrm{d}x = 0$$

We are then left with the result that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos mx\ \mathrm{d}x \quad (m = 1, 2, \dots) \tag{4}$$

Note that in view of (3), we can also use (4) for $n = 0$. We can find the $b_n$ coefficients in a similar way. We go back to (2) and multiply by $\sin mx$ and integrate. This time the first two terms on the RHS vanish, leaving

$$\int_{-\pi}^{\pi} f(x) \sin mx\ \mathrm{d}x = \sum_{n=1}^{\infty} b_n \underbrace{\int_{\pi}^{\pi} \sin nx \sin mx\ \mathrm{d}x}_{\pi\delta_{mn}} = \pi b_m$$

and hence

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\ \mathrm{d}x \quad (m = 1, 2, \dots) \tag{5}$$

Note that if $f(x)$ is even about $x = 0$ then the integrand in (5) is odd, and hence $b_n = 0$. Similarly if $f(x)$ is odd about $x = 0$ then $a_n == 0$ (including $a_0$). These results can be quoted when appropriate.

It can be seen that we only require $f(x)$ to be integrable over the range $-\pi$ to $\pi$. This means that $f$ could be discontinuous and still have a Fourier series representation. In fact all we require is that it is piecewise continuous, i.e. it has a finite number of finite discontinuities. It is also important to realise that provided you only want to represent a function by a Fourier series over a specific range, the function itself need not be periodic as the next example shows.

**Example 2.4.** Obtain the Fourier series for the function $\pi^2 - x^2$ over the range $-\pi \leq x \leq \pi$. We write

$$\pi^2 - x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

and note that the RHS is a periodic function of period $2\pi$. Using the formulae for the Fourier coefficients derived above:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\ \mathrm{d}x = \cdots = 4\pi^2/3$$

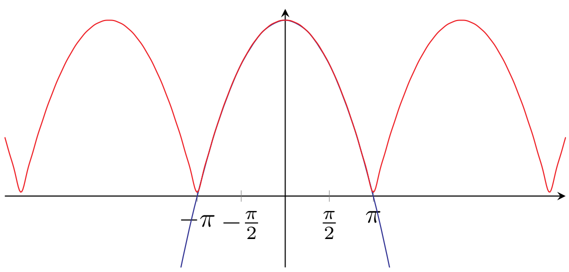
$$a_n = \frac{1}{\pi} \cos nx\ \mathrm{d}x = \cdots - \frac{2}{\pi n^2}[x \cos nx]_{\pi}^{\pi} = 4(-1)^{n+1}/n^2$$

where we have integrated by parts twice, and used that $\sin nx = 0$ and $\cos nx = (-1)^n$. Since $\pi^2 - x^2$ is even about $x = 0$ we can say without further calculation that $b_n = 0$ for all $n$. We therefore have the result that

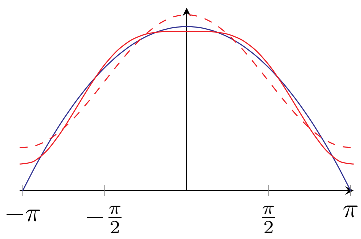
$$\begin{aligned}\pi^2 - x^2 &= \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2} \\ &= \frac{2\pi^2}{3} + 4\cos x - \cos 2x + \ldots\end{aligned}$$

For $-\pi \leq x \leq x$. We can see that the infinite series converges absolutely by comparison with $\sum_{n=1}^{\infty}(1/n^2)$.

In the picture below we show the original parabola (blue) and the Fourier series with an infinite number of terms (red)




We also show the approximations resulting from retaining a finite number of terms (red = 3 terms, dashed = 2) - these are known as *partial sums*:

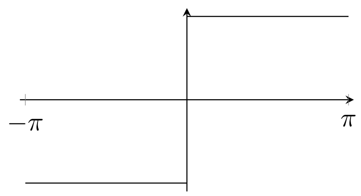



Although the original function is a smooth parabola, the Fourier series, when extended outside $[-\pi, \pi]$ is not smooth, as it is of course a periodic extension of its form within $[-\pi, \pi]$.

*Exercise (non-smooth function):* Obtain the Fourier series for the function $f(x) = |x|$ over the range $[-\pi, \pi]$.

**Example 2.5** (discontinuous function)**.** Derive the Fourier series for the square wave Lecture 18

$$f(x) = \begin{cases} -1 & (-\pi < x < 0) \\ +1 & (0 < x < \pi) \end{cases}$$




we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \sin nx}_{\text{even about } x=0} \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} (1) \sin nx \, \mathrm{d}x \end{aligned}$$

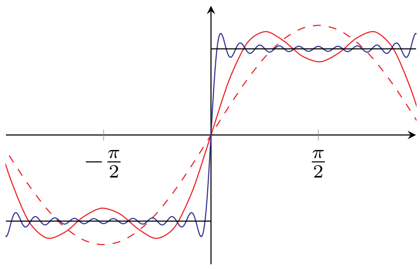
since the integrand is even about $x = 0$, it follows that

$$\begin{aligned} b_n &= \frac{2}{n\pi} [-\cos nx]_0^{\pi} \\ &= \frac{2}{n\pi} ((-1)^{n+1} + 1) \\ &= \begin{cases} 0 & n \text{ even} \\ 4/n\pi & n \text{ odd} \end{cases} \end{aligned}$$

Thus we can express the square wave as

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right), \quad (-\pi < x < \pi) \qquad (6)$$

Plot of Fourier series with one (dotted), three (red) and ten (blue) terms:




## Calculating infinite sums using Fourier series

We can obtain interesting results by substituting in various values of $x$. For example if we consider the square wave example and choose $x = \pi/2$, we know that $f(x) = 1$ from the original definition of $x$, and hence, putting $x = \pi/2$ into the series:

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \dots\right)$$

Rearranging, we conclude:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

(We can check that the series on the right converges by using the alternating series test). In the parabola example we can substitute $x = 0$ and $x = \pi$ and obtain the results

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

## The behaviour of Fourier series at a discontinuity

In the previous example, the square wave is discontinuous at $x = 0$ with $\lim_{x\to 0+} f(x) = 1$, and $\lim_{x\to 0-} f(x) = -1$. We see, from (6), that the value of the Fourier series is zero, i.e. the average of the right hand and left hand limits. This result is true in general as we shall see shortly. The Fourier series (6) also has the value zero at $x = \pm\pi$. At these values of $x$ there is a discontinuity in the periodically-extended function. Another issue concerns how the partial sums converge to the mean value. In figure 6 we plot the partial sums for (6) with 1, 2 and 10 terms retained in the series. There is also an animation of this I will show. It can be observed that the partial sums do not converge smoothly to the mean value at a point of discontinuity. This is known as *Gibbs phenomenon*, which we will discuss in more detail shortly.

## 2.3 Convergence of Fourier series

We would like to study what happens to the Fourier series for $f(x)$ as we include more and more terms. How do we approach the function $f(x)$ and what happens if $f(x)$ is discontinuous as in the previous example? To do this we need to use the following theorem, which we quote without proof.

**Lemma 2.6** (Riemann)**.** *If $g(x)$ is an integrable function in the interval $[a, b]$, then*

$$\lim_{\alpha\to\infty} \int_a^b g(x) \sin \alpha x \, \mathrm{d}x = 0$$

*Proof.* Omitted. Easy to see if $g(x)$ is a constant.

### Behaviour of the partial sums

We now consider the partial sums of a given Fourier series, $S_N(x)$, defined over $-\pi < x < \pi$ as

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \cos nu \, \mathrm{d}u, \quad b_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(y) \sin nu \, \mathrm{d}u$$

We will suppose that the function $f(x)$ may have finite discontinuities at certain values of $x$. At such discontinuities we define the left-hand and right-hand limits as

$$f(x-) = \lim_{\varepsilon\to 0-} f(x+\varepsilon), \; f(x+) = \lim_{\varepsilon\to 0+} f(x+\varepsilon),$$

(Of course, if there is no discontinuity then these limits are equal). We will also assume that the left-hand and right-hand derivatives exist at these points (see figure 7).

Turning now to the expression for $S_N$ , substituting for the coefficients, interchanging the order of summation and the integration, and using $\cos nx \cos nu + \sin nx \sin nu \equiv \cos n(ux)$ we get

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \left[\frac{1}{2} + \sum_{n=1}^{N} \cos n(u-x)\right] \mathrm{d}u$$

It can be shown that

$$1 + 2\sum_{n=1}^{N} \cos n\theta = \frac{\sin[(N+1/2)\theta]}{\sin(\theta/2)}$$

[PS #5]. We can therefore rewrite our expression for the partial sum as

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{\pi}^{\pi} f(u) \frac{\sin[(N+\frac{1}{2})(u-x)]}{\sin[\frac{1}{2}(u-x)]} \, \mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(x+s) \frac{\sin[(N+\frac{1}{2})s]}{\sin\frac{1}{2}s} \, \mathrm{d}s \end{aligned}$$

upon making the substitution $s = ux$.

We want to investigate the behaviour of $S_N(x)$ as $N \to \infty$. We cant use Riemanns lemma directly because $g(s) = f(x+s)/\sin(\frac{1}{2}s)$ is not integrable over a range containing $s = 0$. We will therefore split the range of integration to deal with the problem near $s = 0$ separately, and we also let $\alpha = N + 1/2$. For some $\delta > 0$, we write

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\left(\int_{\pi-x}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} \int_{\delta}^{\pi-x}\right) f(x+s)\frac{\sin\alpha s}{\sin\frac{1}{2}s}\ \mathrm{d}s \\ &\equiv I_1 + I_2 + I_3 + I_4 \end{aligned}$$

The integrals $I_1$ and $I_4$ tend to zero by Riemann's lemma. Now consider the second integral:

$$\begin{aligned} I_2 = \frac{1}{2\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{\sin\frac{1}{2}s}\ \mathrm{d}s \equiv \frac{1}{\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{s}\ \mathrm{d}s \\ + \frac{1}{2\pi}\int_{-\delta}^{0} f(x+s)\left\{\frac{1}{\sin\frac{1}{2}s} - \frac{2}{s}\right\}\sin\alpha s\ \mathrm{d}s \end{aligned} \tag{7}$$

The second integral on the right hand side in (7) tends to zero as $\alpha \to \infty$ by Riemanns lemma, as the term in curly brackets is well-behaved at $s = 0$. Turning to the first integral on the right-hand-side of (7):

$$\begin{aligned} \frac{1}{\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{s}\ \mathrm{d}s \equiv \frac{1}{\pi}\int_{-\delta}^{0}\left\{\frac{f(x+s)-f(x-)}{s}\right\}\sin\alpha s\ \mathrm{d}s \\ + \frac{f(x-)}{\pi}\int_{-\delta}^{0}\frac{\sin\alpha s}{s}\ \mathrm{d}s \end{aligned} \tag{8}$$

Again, the term in curly brackets is well-behaved at $s = 0$ because as $s \to 0-$ this term is simply the left-hand derivative which we are assuming is well-defined. Therefore the first term on the right-hand-side of (8) tends to zero by Riemanns lemma.

We are therefore left with

$$\begin{aligned} \lim_{\alpha\to\infty} I_2 &= \frac{f(x-)}{s}\lim_{\alpha\to\infty}\int_{-\delta}^{0}\frac{\sin\alpha s}{s}\ \mathrm{d}s \\ &= \frac{f(x-)}{s}\lim_{\alpha\to\infty}\int_{\alpha\delta}^{0}\frac{-\sin q}{-q/\alpha}\frac{\mathrm{d}q}{(-\alpha)} \\ &= \frac{(x-)}{\pi}\int_{0}^{\infty}\frac{\sin q}{q}\ \mathrm{d}q \\ &= \frac{1}{2}f(x-) \end{aligned}$$

letting $q = -\alpha s$, since the standard integral $\int_0^\infty (\sin q)/q\ \mathrm{d}q = \pi/2$. IN a similar way, we can show

$$\lim_{\alpha\to\infty} I_3 = \frac{1}{2}f(x+)$$

Putting this all together:

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}[f(x+) + f(x-)]$$

We have therefore shown that at a point of discontinuity the Fourier series converges to the average value of $f$, which is what we saw explicitly in the square wave example. Indeed this also shows that if $f(x+) = f(x-)$, i.e. the function is continuous at $x$, then the Fourier series is $f(x)$.

## Gibbs phenomenon

From our computations of the Fourier series for the square wave (pg. 60) we saw that the series behaves strangely near $x = 0$, overshooting the required jump by the same amount for all $N$. Lets examine the specific series in more detail. Recall that the Fourier series in question is Thus we can express the square wave as Lecture 19

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right), \quad (-\pi < x < \pi)$$

The sum of the first $N$ terms is

$$S_N(x) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{\sin[(2n-1)x]}{2n-1}$$

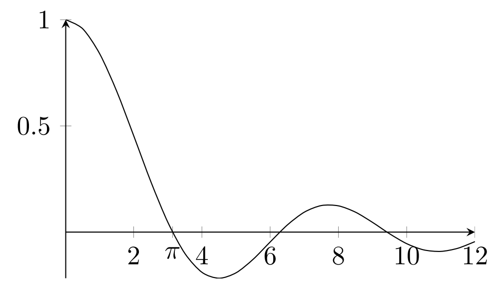
and this can be rewritten in the form

$$S_N(x)\frac{2}{\pi}\int_0^x \frac{\sin[(2n-1)\theta]}{2n-1}\,\mathrm{d}\theta$$

(see PS #5). Making the substitution $\xi = 2N\theta$ and assuming $N >> 1$, we find

$$S_N(x) = \frac{2}{\pi}\int_0^{2Nx} \frac{\sin\xi}{2N\sin(\xi/2N)}\,\mathrm{d}\xi \sim \frac{2}{\pi}\int_0^{2Nx} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi$$

If we fix $x > 0$ and let $N \to \infty$ we see that $S_N(x) \to 1$ (recall $\int_0^\infty (\sin x)/x\,\mathrm{d}x = \pi/2$). This is what we expect since the Fourier series is representing the function $f(x) = 1$ when $x > 0$. Similarly if we fix $x < 0$ and let $N \to \infty$ we get $S_N(x) \to -1$. Now lets find the value of x at which $S_N(x)$ is maximum. If we think about the area under the curve $\sin\xi/\xi$:



We see that the maximum value of the integral occurs when $2Nx = \pi$. At this point we can compute

$$\int_0^\pi (\sin x)/x \ \mathrm{d}x \approx 1.82$$
$$\implies S_N(\pi/2N) \approx (2/\pi)(1.852) \approx 1.179$$

a value which is independent of $N$. This means there is always an overshoot (by a factor 1.179) but that overshoot occurs closer and closer to the discontinuity at $x = 0$ as we increase the number of terms in the Fourier series.

**Theorem 2.7: Parseval**

If $f(x)$ is represented by a Fourier series as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} \quad (-\pi \leq x \leq \pi)$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 \ \mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\{a_n^2 + b_n^2\}$$

*Proof.* We write $[f(x)]^2$ as

$$[f(x)]^2 = \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]\left[\frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx\right]$$

noting the $m$ index in the second factor. Expanding this out, we get

$$= \frac{1}{4}a_0^2 + \frac{1}{2}a_0 \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx + \frac{1}{2}a_0 \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_m a_m \cos nx \cos mx + b_n b_m \sin nx \sin mx + a_n b_m \cos nx \sin mx$$
$$+ a_m b_n \cos mx \sin nx]$$

Now we integrate both sides of this expression with respect to $x$ between $-\pi$ and $\pi$. Most of the terms integrate to zero, leaving

$$\int_{\pi}^{\pi} [f(x)]^2 \ \mathrm{d}x = \frac{\pi}{2}a_0^2 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_n a_m \pi \delta_{mn} + b_n b_m \pi \delta_{mn})$$
$$= \frac{\pi}{2}a_0^2 + \sum_{n=1}^{\infty}\pi(a_n^2 + b_n^2)$$

Hence the result. ∎

**Example 2.8.** *Compute the Fourier series for* $\cos(x/2)$ *over* $[-\pi, \pi]$. *Use Parseval's theorem to compute the value of*

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}$$

First we observe that $\cos(x/2)$ is even about $x = 0$, hence $b_n = 0$ for all $n$. We can therefore write

$$\cos\left(\frac{x}{2}\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad (-\pi \leq x \leq \pi)$$

The Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x/2)\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{2\pi} \cos(x/2)\ \mathrm{d}x \\ &= \frac{2}{\pi}\left[2\sin(x/2)\right]_0^{\pi} = 4/\pi \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} \cos(x/2)\cos(nx)\ \mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos[(n+\tfrac{1}{2})x] + \cos[(n-\tfrac{1}{2})x]\ \mathrm{d}x \\ &= \frac{1}{\pi}\left(\frac{\sin[(n+\frac{1}{2})\pi]}{n+\frac{1}{2}} + \frac{\sin[(n-\frac{1}{2})\pi]}{n-\frac{1}{2}}\right) \end{aligned}$$

Note that $\sin(n+\frac{1}{2})\pi = 2\cos n\pi \sin\frac{\pi}{2} = 2(-1)^n$, so

$$\begin{aligned} &= \frac{(-1)^n}{\pi}\left(\frac{2}{2n+1} + \frac{2}{2n-1}\right) \\ &= \frac{4(-1)^{n+1}}{\pi(4n^2-1)} \end{aligned}$$

Therefore, using Parseval's Theorem:

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} [\cos(x/2)]^2\ \mathrm{d}x &= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 \\ &= \frac{8}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} \end{aligned}$$

But the LHS is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \frac{1}{2}\cos x\right)\ \mathrm{d}x = 1$$

Therefore we conclude that

$$\frac{8}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} = 1$$

and hence

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} = \frac{\pi^2-8}{16}$$

## 2.4 Fourier Series over a general interval

The use of Fourier series would obviously be limited if it were confined to the range $[-\pi, \pi]$. However the theory is easily generalised to an arbitrary interval by observing that $\sin(n\pi x/L), \cos(n\pi x/L)$ have period $2L/n$ and the set of functions

$$\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, \ldots,)$$

are orthonormal over the interval $[a, a+2L]$ where $a$ is any real number, i.e. we have

$$\int_a^{a+2L}\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0$$

$$\int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = 0$$

$$\int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = \int_a^{a+2L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = L\delta_{mn}$$

which agrees with the previous results when $L = \pi$. Using these results, we can then represent a function $f(x)$ over the interval $[a, a+2L]$ in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

By proceeding in a similar fashion to $[-\pi, \pi]$ case, we can establish that the corresponding Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_a^{a+L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{L}\int_a^{a+L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad n = 1, 2, \ldots$$

Obviously, these formulae reduce to the $[-\pi, \pi]$ case when $a = \pi$ and $L = \pi$. It follows that if $f(x)$ is integrable over a finite interval, a Fourier series can be found for $f(x)$ in this interval.

Its straightforward to generalise Parsevals result for the $[\pi, \pi]$ interval to the interval $[a, a+2L]$:

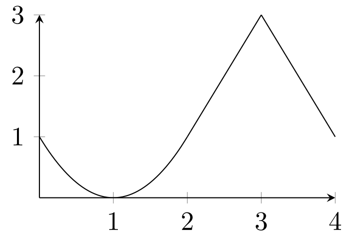
**Theorem 2.9: Parseval for Fourier series over general interval**

$$\frac{1}{L}\int_a^{a+2L} [f(x)]^2 \,\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

**Example 2.10.** Find the Fourier expansion of the periodic function whose definition over one period is Lecture 20

$$f(x) = \begin{cases} (x-1)^2 & (0 \le x \le 2) \\ 2x-3 & (2 \le x \le 3) \\ 9-2x & (3 \le x \le 4) \end{cases}$$

The function is sketched below




First we see that the period $2L = 4$ in this case, and so $L = 2$. We therefore use our formulae with $a = 0$ and $L = 2$, so that

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^4 f(x) \,\mathrm{d}x \\ &= \frac{1}{2}\int_0^2 (x-1)^2 \,\mathrm{d}x + \int_2^3 (2x-3) \,\mathrm{d}x + \int_3^4 (9-2x) \,\mathrm{d}x \\ &= \cdots = 7/3 \end{aligned}$$

Similarly

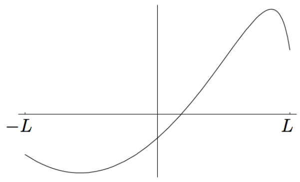
$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\left(\frac{n\pi x}{2}\right) \mathrm{d}x \qquad b_n = \frac{1}{2}\int_0^4 f(x)\sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x$$

After some algebra, we obtain

$$a_n = \frac{8}{n^2\pi^2}\cos\left(\frac{3n\pi}{2}\right) \qquad b_n = \frac{8}{n^3\pi^3}(n\pi\sin\left(\frac{3n\pi}{2}\right) - 1 + \cos n\pi)$$

## 2.5 Half-range Fourier series

The series we have looked at so far are known as *full-range Fourier series* in view of the fact that the function is represented by the Fourier series over one full period of the series. Suppose we have a function $f(x)$ defined over the range $[-L, L]$, e.g.

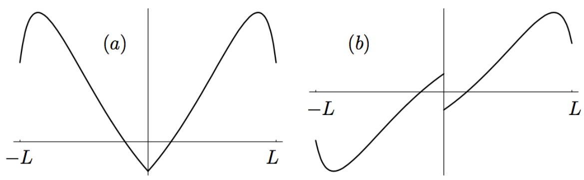



Now consider the following two real functions defined over the same interval

$$f_1(x) = \begin{cases} f(x) & (0 \le x \le L) \\ f(-x) & (-L \le x \le 0) \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & (0 < x < L) \\ -f(-x) & (-L < x < 0) \end{cases}$$

The functions are sketched in below ($f_1(x)$ in (a), $f_2(x)$ in (b))




Clearly $f_1(x)$ is even about $x = 0$ and hence the Fourier coefficients $b_n = 0$ in its Fourier series expansion over $[-L, L]$, i.e.

$$f_1(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad (-L \le x \le L)$$

and

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2}{L}\int_{0}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n = 0, 1, 2, \ldots,) \end{aligned}$$

with the last line following from the evenness of the integrand about $x = 0$. In contrast, the function $f_2(x)$ is odd about $x = 0$ and hence the Fourier expansion is simply

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (-L < x < L)$$

(note that the interval is open due to the oddness of $f_2(x)$ about $x = 0$).

The coefficient of $b_n$ are:

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2}{L}\int_{0}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n = 1, 2, \ldots,) \end{aligned}$$

Now note that by definition, both $f_1$ and $f_2$ are equal over the range $(0, L)$. There are therefore two ways of representing $f$ over this interval:

$$(i) \quad f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad (0 \le x \le L)$$
$$a_n = \frac{2}{L}\int_0^L f_1(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n = 0, 1, 2, \ldots,)$$

which is known as the *half-range Fourier cosine series*, and

$$(ii) \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (0 < x < L)$$
$$b_n = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n = 1, 2, \ldots,)$$

which is the *half-range Fourier sine series.*

Again, analogous results to Parsevals formula can be found for half-range series.

**Theorem 2.11: Parseval for half-range series**

$$\frac{2}{L}\int_0^L [f(x)]^2\,\mathrm{d}x = \begin{cases} \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 & \text{(Fourier cosine series)} \\ \sum_{n=1}^{\infty} b_n^2 & \text{(Fourier sine series)} \end{cases}$$

**Example 2.12.** Find the half-range Fourier cosine and Fourier sine series for $f(x) = x$ over the range $0 < x < \pi$.

Let's start with the cosine series. Since the range is 0 to $\pi$, we put $L = \pi$ in our formulae to get

$$x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

The Fourier coefficients can be calculated as

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \pi \\ a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\left\{\underbrace{\left[\frac{x}{n}\sin nx\right]_0^\pi}_{zero} - \int_0^\pi \frac{1}{n}\sin nx\,\mathrm{d}x\right\} \\ &= \frac{2}{n^2\pi}(\cos n\pi - 1) \end{aligned}$$

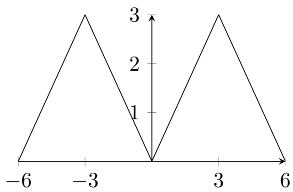
Therefore we write

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2}$$

or putting $n = 2m - 1$:

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos[(2m-1)x]}{(2m-1)^2} \qquad (0 \leq x \leq \pi)$$

note that the end points are included, because this is the even extension of $x$ outside the range $[0, \pi]$. See picture:




Now let's derive the Fourier sine series expansion. We write

$$x = \sum_{n=1}^{\infty} b_n \sin nx$$

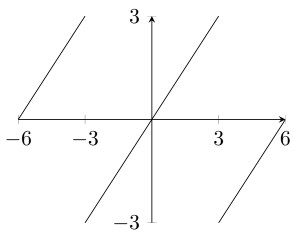
where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, \mathrm{d}x \\ &= \ldots \text{(by parts)} \\ &= \frac{2}{n}(-1)^{n+1} \end{aligned}$$

Thus we have the result

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx \quad (0 \leq x < \pi) \tag{9}$$

Since this is the odd extension of $x$ outside the range 0 to $\pi$, it is generally only valid in the open interval $(0, \pi)$. However because $f(0) = 0$ in this case, the representation is also clearly valid at $x = 0$, and so the region of validity of (9) is $0 \leq x < \pi$. The series is shown below over the range $[-2\pi, 2\pi]$.



## 2.6 Integration and differentiation of Fourier series

In our previous example on half-range series, we say that Lecture 21

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx, \quad (0 \leq x < \pi) \tag{10}$$

We can integrate both sides of (10) with respect to x and get the correct answer provided our range of integration lies inside the region of validity of the original series (0 to $\pi$ in this case). For example, provided $0 < X < \pi$ we can write

$$\int_0^X x \, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \int_0^X \sin nx \, \mathrm{d}x$$

$$\implies \frac{1}{2}X^2 = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}(\cos nx - 1)$$

The conditions under which we can differentiate are more restrictive. (This is not surprising since differentiating a Fourier series brings out an extra factor $n$, making the series much less likely to converge). To be able to differentiate we need the function, when extended periodically, to be continuous for all $x$. This is not true for the series in (10). If we decide to differentiate both sides anyway we get

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx \quad \text{WRONG!}$$

This cannot be correct as the general term in the series does not tend to zero and hence the series must diverge. If however, we start with the half-range cosine series representation for $x$:

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos[(2m-1)x]}{(2m-1)^2} \quad (0 \leq x \leq \pi)$$

then this is continuous when extended periodically, and so we can differentiate to get

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2m-1)x]}{(2m-1)}$$

which *is* a valid result, provided $0 < x < \pi$. [We can't differentiate a second time though]

## 2.7 Exponential form of Fourier series

This alternative representation can sometimes simplify calculations. It is used frequently in engineering applications and writing the formulae in this way provides a clear link to Fourier transforms which we will explore in the next section of the course. Recalling that

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin nx = -\frac{i}{2}(e^{inx} - e^{-inx})$$

we can write

$$a_n \cos nx + b_n \sin nx = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$$

Therefore we can rewrite our Fourier series representation over $[-\pi, \pi]$ in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n e^{inx} + k_n e^{-inx} \tag{11}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x)\cos nx - if(x)\sin nx)\ \mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ \mathrm{d}x \quad (n = 1, 2, \ldots,) \end{aligned}$$

Also note that $c_0 = a_0/2$. Similarly:

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\ \mathrm{d}x \quad (n = 1, 2, \ldots,)$$

If we write $c_{-n} = k_n$, we can express this more succinctly as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad |x| < \pi \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ \mathrm{d}x \quad (n = 0, \pm 1, \pm 2, \ldots,) \end{aligned}$$

For a function of period $2L$ this is easily generalised to

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad |x| < L \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\ \mathrm{d}x \quad (n = 0, \pm 1, \pm 2, \ldots,) \end{aligned}$$

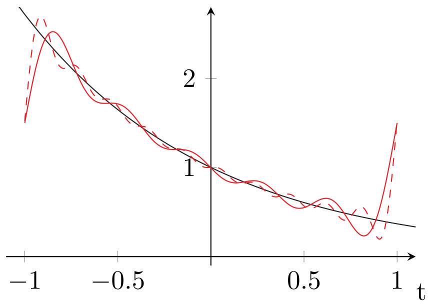
**Example 2.13.** Find the complex form of the Fourier series of the periodic function whose definition over one period is $f(x) = e^{-x}$, $-1 < x < 1$. Since $L = 1$ we have

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\pi x}\ \mathrm{d}x \\ &= -\frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{1 + in\pi}\right]_{-1}^{1} \\ &= -\frac{1}{2}(e^{-1} - e^{1})\frac{\cos n\pi}{1 + in\pi} \\ &= (-1)^n \sinh(1)\frac{(1 - in\pi)\cos n\pi}{1 + n^2\pi^2} \end{aligned}$$

Therefore the complex Fourier series representation is

$$e^{-x} = \sum_{n=-\infty}^{\infty} (-1)^n (1 - in\pi) \sinh(1) \frac{e^{in\pi x}}{1 + n^2\pi^2}$$

The function $e^{-x}$ and the partial sums of the series with 5 terms (solid red) and 10 terms (dashed red) are shown together below. Note the Gibbs phenomenon near $x = \pm 1$ where the series converges to $\frac{1}{2}(e + e^{-1}) = \cosh(1)$.





“I'm not going to say the word I'm thinking of.”

# 3 Fourier Tranforms

A key reason for studying this material (and Fourier series) is that we can use these ideas to help us solve certain partial differential equations which we will study in the next section. However there are many other uses for Fourier transforms and they are used particularly by scientists and engineers in the context of signal processing. We have seen that Fourier series allows us to represent a given function in terms of sine and cosine waves of different amplitudes and frequencies, but this representation is only valid over a finite range of the independent variable. We now wish to study what happens if we take a Fourier series defined over $[L, L]$ and let $L \to \infty$ First we write out the Fourier series over the finite range. It is convenient to use the exponential form derived at the end of the previous section:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad |x| < L$$
$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, \mathrm{d}x \quad (n = 0, \pm 1, \pm 2, \dots)$$

**Definition.** We define the angular frequency

$$\omega_n = n\pi/L$$

and also the frequency difference

$$\delta\omega = \omega_{n+1} - \omega_n = \pi/L$$

In terms of this new notation, the Fourier series becomes

$$f(x) = \frac{1}{2\pi} \left\{ \sum_{n=-\infty}^{\infty} \left\{ \int_{-L}^{L} f(s) e^{-i\omega_n s} \, \mathrm{d}s \right\} e^{i\omega_n x} \right\} \delta\omega$$

Now as $L \to \infty$, $\delta\omega \to 0$ and

$$\sum_{n=-\infty}^{\infty} g(\omega_n) \, \delta\omega \to \int_{-\infty}^{\infty} g(\omega) \, \mathrm{d}\omega$$

(think about splitting the integral up into strips of width $\delta\omega$.) So in the limit $L \to \infty$ we obtain the result that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(s) e^{-i\omega s} \, \mathrm{d}s \right\} e^{i\omega x} \, \mathrm{d}\omega$$

We have therefore shown that for a function $f(x)$ defined over $-\infty < x < \infty$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, \mathrm{d}\omega \tag{12}$$

where we define:

**Definition.**

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, \mathrm{d}x$$

is the *Fourier Transform of f*

**Notation.** Fourier transform:

$$\hat{f}(\omega) = \mathcal{F}\{f(x)\} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, \mathrm{d}x$$

Inverse Fourier transform:

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x} \, \mathrm{d}\omega$$

**Theorem 3.1: Fourier's integral formula**

This is the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(s)e^{-i\omega s} \, \mathrm{d}s \right\} e^{i\omega x} \, \mathrm{d}\omega$$

that we arrived at in a non-rigorous way in the previous section.

*Proof.* To prove this we need to assume that $f(x)$ is such that

$$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x$$

converges. We will also assume that $f(x)$ and $f'(x)$ are continuous for all $x$, although this condition can be relaxed as we will discuss at the end. We start by writing the RHS above in the form

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \left\{ \int_{-\infty}^{\infty} f(s)e^{-i\omega s} \, \mathrm{d}s \right\} e^{-i\omega x} \, \mathrm{d}\omega \\
&= \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \left\{ \int_{-\infty}^{\infty} f(s)e^{-i\omega(s-x)} \, \mathrm{d}s \right\} \mathrm{d}\omega \\
&= \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \left[ \left\{ \int_{-\infty}^{\infty} f(s)\cos[\omega(s-x)] \, \mathrm{d}s \right\} - i \left\{ \int_{-\infty}^{\infty} f(s)\sin[\omega(s-x)] \, \mathrm{d}s \right\} \right] \mathrm{d}\omega
\end{aligned}$$

The first integral in curly brackets is even about $\omega = 0$, while the second is odd. The expression therefore simplifies to

$$\lim_{L\to\infty} \frac{1}{\pi} \int_{0}^{L} \left\{ \int_{-\infty}^{\infty} f(s)\cos[\omega(s-x)] \, \mathrm{d}s \right\} \mathrm{d}\omega$$

Because of the absolute convergence of the inner integral we can interchange the order of integration and write this as Lecture 22

$$\begin{aligned}
&\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\int_0^L \cos[\omega(s-x)]\ \mathrm{d}\omega\right\} f(s)\ \mathrm{d}s\\
&=\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\frac{\sin[L(s-x)]}{s-x}\ \mathrm{d}s\\
&=\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin(Lu)}{u}\ \mathrm{d}u
\end{aligned}$$

using the substitution $u = s - x$.

We now split the integral into two parts in the form

$$\lim_{L\to\infty}\frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\left(\frac{f(x+u)-f(x)}{u}\right)\sin(Lu)\ \mathrm{d}u + f(x)\int_{-\infty}^{\infty}\frac{\sin(Lu)}{u}\ \mathrm{d}u\right\}$$

The first integral tends to zero as $L \to \infty$ using Riemann's lemma. We then use the substitution $p = Lu$ in the second integral to leave

$$\lim_{L\to\infty}\frac{1}{\pi}f(x)\int_{-\infty}^{\infty}\frac{\sin(p)}{p/L}\frac{\mathrm{d}p}{L} = f(x)$$

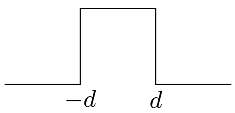
using the fact that $\int_{-\infty}^{\infty}(\sin p)/p\ \mathrm{d}p = \pi$ (Problem sheet 6) ∎

As remarked earlier, we have assumed here that $f(x)$ is continuous at all $x$. If there is a discontinuity at $x_0$ (with finite left and right derivatives there), the LHS of the formula is replaced by $[f(x_0+) + f(x_0-)]/2$ (analogous to the Fourier series convergence we investigated earlier).

**Example 3.2.** Find the fourier transform of the square wave

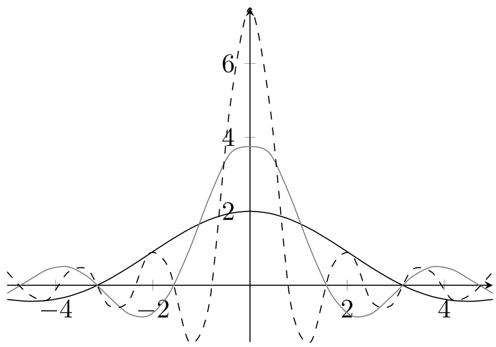
$$f(x) = \begin{cases} 1, & |x| < d \\ 0, & |x| > d \end{cases}$$

We say $f$ has "compact support", it's non-zero on a finite support:




We have that

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\ \mathrm{d}x\\
&= \int_{-d}^{d} e^{-i\omega x}\ \mathrm{d}x = \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-d}^{d}\\
&= -\frac{1}{i\omega}\left\{e^{-\omega d} - e^{i\omega d}\right\} = \frac{2}{\omega}\sin(\omega d)
\end{aligned}$$




Note that as $d$ gets larger, $\hat{f}$ becomes more concentrated in the vicinity of $\omega = 0$.

*Exercise:* Find the Fourier transform of the function

$$f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(Answer: $1/(a + i\omega)$ )

## 3.1 Fourier cosine and sine tranforms

In the same way that we exploited symmetry to define half-range Fourier series, we can similarly define transforms over the range $[0, \infty)$. First if we suppose that $f(x)$ is even about $x = 0$ we can write

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \underbrace{f(x) \cos \omega x}_{\text{even about } x=0} - \underbrace{i f(x) \sin \omega x}_{\text{odd about } x=0} \, \mathrm{d}x \\ &= 2 \int_{0}^{\infty} f(x) \cos \omega x \, \mathrm{d}x \end{aligned}$$

**Definition.** We define

$$\hat{f}_c(\omega) = \int_0^{\infty} f(x) \cos \omega x \, \mathrm{d}x$$

to be the *Fourier cosine transform.*

so for an even function $f(x)$:

$$\hat{f}(\omega) = 2 \hat{f}_c(\omega)$$

Note that $\hat{f}_c(\omega)$ is even about $\omega = 0$. Using the inversion formula for the regular tranform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, \mathrm{d}\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{f}_c(\omega) e^{i\omega x} \, \mathrm{d}\omega$$

Exploiting the evenness of $\hat{f}_c(\omega)$, this reduces to

$$f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_c(\omega)\cos\omega x \ \mathrm{d}\omega$$

which is the *inversion formula for the Fourier cosine transform.*

In a similar way, by considering $f(x)$ to be odd about $x = 0$, we can define a Fourier sine transform and derive the corresponding inversion formula. We obtain the pair of expressions

$$\begin{aligned}\hat{f}_s(\omega) &= \int_0^\infty f(x)\sin\omega x \ \mathrm{d}\omega \\ f(x) &= \frac{2}{\pi}\int_0^\infty \hat{f}_s(\omega)\sin\omega x \ \mathrm{d}\omega\end{aligned}$$

## 3.2 Properties of Fourier transforms

**(i)** The Fourier transform in linear, and so

$$\mathcal{F}\{af(x) + bg(x)\} = a\hat{f}(\omega) + b\hat{g}(\omega)$$

where $a$ and $b$ are constants. It follows from this that

$$\mathcal{F}^{-1}\{a\hat{f}(\omega) + b\hat{g}(\omega)\} = af(x) + bg(x)$$

where $\mathcal{F}^{-1}$ denotes the inverse transform.

**(ii)** If $a > 0$:

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$$

*Proof.* Starting on the LHS and making the substitution $s = ax$:

$$\begin{aligned}\mathcal{F}\{f(ax)\} &= \int_{-\infty}^\infty f(ax)e^{-\omega x} \ \mathrm{d}x \\ &= \frac{1}{a}\int_{-\infty}^\infty f(s)e^{i(\omega/a)s} \ \mathrm{d}s \\ &= \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)\end{aligned}$$

∎

**(iii)** In a similar way, we can establish that

$$\mathcal{F}\{f(-x)\} = \hat{f}(-\omega)$$

**(iv)** The transform of a shifted function can be calculated as follows (using $s = x - x_0$) Lecture 23

$$\begin{aligned}\mathcal{F}\{f(x - x_0)\} &= \int_{-\infty}^\infty f(x - x_0)e^{-i\omega x} \ \mathrm{d}x \\ &= \int_{-\infty}^\infty f(s)e^{i\omega(s+x_0)} \ \mathrm{d}s \\ &= e^{-\omega x_0}\hat{f}(\omega)\end{aligned}$$

**(v)** This is a similar result, but this time involving a shift in transform space:

$$\begin{aligned}
\mathcal{F}\{e^{i\omega_0 x} f(x)\} &= \int_{-\infty}^{\infty} f(x) e^{i\omega_0 x} e^{-i\omega x}\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x) e^{-i(\omega-\omega_0)x}\ \mathrm{d}x \\
&= \hat{f}(\omega - \omega_0)
\end{aligned}$$

**(vi)** Symmetry formula. The following result is surprisingly useful. Suppose the Fourier transform of $f(x)$ is $\hat{f}(\omega)$; change the variable $\omega$ to $x$, then

$$\mathcal{F}\{\hat{f}(x)\} = 2\pi f(-\omega)$$

*Proof.* Starting with the inversion formula, we have

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\ \mathrm{d}\omega \\
&\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\smiley) e^{i\smiley x}\ \mathrm{d}\smiley
\end{aligned}$$

If we now let $x = -\omega$:

$$\begin{aligned}
f(-\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\smiley) e^{-i\omega\smiley}\ \mathrm{d}\smiley \\
&\equiv \frac{1}{2\pi}\hat{f}(x) e^{-i\omega x}\ \mathrm{d}x \\
&= \frac{1}{2\pi}\mathcal{F}\{\hat{f}(x)\}
\end{aligned}$$

as required. ∎

The following results are particularly useful when applying Fourier transforms to partial differential equations.

**(vii)**

$$\mathcal{F}\{d^n f/dx^n\} = (i\omega)^n \hat{f}(\omega)$$

*Proof.* This can be established by integration by parts. We assume that all derivatives of $f$ tend to zero as $x \to \pm\infty$.

$$\begin{aligned}
\mathcal{F}\{d^n f/dx^n\} &= \int_{-\infty}^{\infty} (d^n f/dx^n) e^{-i\omega x}\ \mathrm{d}x \\
&= [(d^{n-1} f/dx^{n-1}) e^{-i\omega x}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega)\frac{d^{n-1}f}{dx^{n-1}} e^{-i\omega x}\ \mathrm{d}x
\end{aligned}$$

Note that the first term tends to zero as $x \to \pm\infty$, leaving

$$\begin{aligned}
&= i\omega\mathcal{F}\{d^{n-1} f/dx^{n-1}\} \\
&= \dots \\
&= (i\omega)^n \mathcal{F}\{f\}
\end{aligned}$$

∎

**(viii)** $\mathcal{F}\{xf(x)\} = i\hat{f}'(\omega)$.

*Proof.* Considering the LHS:

$$\begin{aligned}\int_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}}{\mathrm{d}\omega}(ie^{-i\omega x})\ \mathrm{d}x \\ &= i\frac{\mathrm{d}}{\mathrm{d}\omega}\int_{-\infty}^{\infty} f(x)e^{-\omega x}\ \mathrm{d}x \\ &= i\frac{d}{\mathrm{d}\omega}\hat{f}(\omega)\end{aligned}$$

∎

**(ix)**

(a) $\mathcal{F}_c\{f'(x)\} = -f(0) + \omega\hat{f}_s(\omega)$.

(b) $\mathcal{F}_s\{f'(x)\} = -\omega\hat{f}_c(\omega)$.

(c) $\mathcal{F}_c\{f''(x)\} = -f'(0) - \omega^2\hat{f}_c(\omega)$.

(d) $\mathcal{F}_s\{f''(x)\} = \omega f(0) - \omega^2\hat{f}_s(\omega)$.

*Proof.* We prove (a) and (c) and leave the others as exercises. For (a) we have, integrating by parts:

$$\begin{aligned}\mathcal{F}_c\{f'(x)\} &= \int_0^{\infty} f'(x)\cos\omega x\ \mathrm{d}x \\ &= [f(x)\cos\omega x]_0^{\infty} - \int_0^{\infty}(-\omega)f(x)\sin\omega x\ \mathrm{d}x \\ &= -f(0) + \omega\hat{f}_s(\omega)\end{aligned}$$

as required, while for (c):

$$\begin{aligned}\mathcal{F}_c\{f''(x)\} &= \int_0^{\infty} f''(x)\cos\omega x\ \mathrm{d}x \\ &= [f'(x)\cos\omega x]_0^{\infty} + \omega\int_0^{\infty} f'(x)\sin\omega x\ \mathrm{d}x \\ &= -f'(0) + \omega\mathcal{F}_s\{f'(x)\} \\ &\text{Then using (b)} \\ &= -f'(0) - \omega^2\hat{f}_c(\omega)\end{aligned}$$

∎

**(x)** If $f(x)$ is a complex-valued function and $[f(x)]^*$ is its complex conjugate, then

$$\mathcal{F}\{[f(x)]^*\} = [\hat{f}(-\omega)]^*$$

*Proof.* We have

$$f(-\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\ \mathrm{d}x \quad (w, x \in \mathbb{R})$$

and so it follows that

$$\begin{aligned}[\hat{f}(-\omega)]^* &= \int_{-\infty}^{\infty} [f(x)]^* e^{-i\omega x} \, \mathrm{d}x \\ &= \mathcal{F}\{[f(x)]^*\}\end{aligned}$$

as required. ■

## 3.3 Convolution theorem for Fourier transforms

**Definition** (Convolution)**.** The convolution of two functions $f(x)$, $g(x)$ defined over $(-\infty, \infty)$, is

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-u)g(u) \, \mathrm{d}u$$

[Note: $f(x) * g(x) = g(*) * f(x)$]. An important result is the so-called convolution theorem:

**Theorem 3.3: Convolution**

$$\mathcal{F}\{f * g\} = \hat{f}(\omega)\hat{g}(\omega)$$

*Proof.* Starting on the LHS, we have

$$\int_{x=-\infty}^{\infty} \left\{ \int_{u=-\infty}^{\infty} f(x-u)g(u) \, \mathrm{d}u \right\} e^{-i\omega x} \, \mathrm{d}x$$

Changing the order of integration:

$$\int_{u=-\infty}^{\infty} g(u) \left\{ \int_{x=-\infty}^{\infty} f(x-u)e^{-i\omega x} \, \mathrm{d}x \right\} \mathrm{d}u$$

Now make the substitution $s = x - u$ at fixed $u$. The double integral becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} g(u) \left\{ f(x-u)e^{-i\omega x} \, \mathrm{d}x \right\} \mathrm{d}u \\ &= \left( \int_{-\infty}^{\infty} g(u)e^{-i\omega u} \, \mathrm{d}u \right) \left( \int_{s=-\infty}^{\infty} f(s)e^{-i\omega s} \, \mathrm{d}s \right) \\ &= \hat{g}(\omega)\hat{f}(\omega)\end{aligned}$$

as required. ■

Note: The convolution is most useful in the form

$$\mathcal{F}^{-1}\{\hat{g}(\omega)\hat{f}(\omega)\} = f * g$$

**Example 3.4.** Find the inverse Fourier transform of the function

$$\frac{1}{(4+\omega^2)(9+\omega^2)}$$

Setting

$$\hat{f}(\omega) = \frac{1}{4+\omega^2} \qquad \hat{g}(\omega) = \frac{1}{9+\omega^2}$$

we have (from problem sheet 6) that

$$f(x) = \frac{1}{4}e^{-2|x|} \qquad g(x) = \frac{1}{6}e^{-3|x|}$$

Thus, by the convolution theorem

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{1}{(4+\omega^2)(9+\omega^2)}\right\} &= f(x) * g(x) \\ &= \frac{1}{24}\int_{-\infty}^{\infty} e^{-2|x-u|}e^{-3|u|}\ \mathrm{d}u \qquad (*) \end{aligned}$$

Considering the integral in $(*)$, assuming $x > 0$:

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-2|x-u|}e^{-3|u|}\ \mathrm{d}u \\ &= \int_{-\infty}^{0} e^{-2|x-u|}e^{3u}\ \mathrm{d}u + \int_{0}^{\infty} e^{-2|x-u|}e^{-3u}\ \mathrm{d}u \\ &= \int_{-\infty}^{0} e^{-2(x-u)}e^{3u}\ \mathrm{d}u + \int_{0}^{\infty} e^{-2|x-u|}e^{-3u}\ \mathrm{d}u \\ &= e^{-2x}\int_{-\infty}^{0} e^{5u}\ \mathrm{d}u + e^{-2x}\int_{0}^{x} e^{-u}\ \mathrm{d}u + e^{2x}\int_{x}^{\infty} e^{-5u}\ \mathrm{d}u \\ &= \frac{1}{5}e^{-2x} - e^{-2x}(e^{-x}-1) + \frac{e^{2x}}{5}e^{-5x} \\ &= \frac{6}{5}e^{-2x} - \frac{4}{5}e^{-3x} \end{aligned}$$

Hence $(*)$ becomes

$$\mathcal{F}^{-1}\left\{\frac{1}{(4+\omega^2)(9+\omega^2)}\right\} = \frac{1}{20}e^{-2|x|} - \frac{1}{30}e^{-3|x|}$$

Note that there are other ways to compute the inverse, e.g. we could decompose the original function into partial fractions and invert term-by-term.

**Theorem 3.5: Energy Theorem**

This is the analogous result to Parseval's theorem for Fourier series. If $f(x)$ is a real-valued function, then Lecture 24

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 = \int_{-\infty}^{\infty}[f(x)]^2 \,\mathrm{d}x$$

*Proof.* Properties (iii) $[\mathcal{F}\{f(-x)\} = \hat{f}(-\omega)]$ and (x) $[\mathcal{F}\{[f(x)]^*\} = [\hat{f}(-\omega)]^*]$ of the Fourier transform give

$$\mathcal{F}\{|f(-x)|^*\} = [\hat{f}(\omega)]^*$$

Since we are assuming $f$ to be real, this simplifies to

$$\mathcal{F}\{f(-x)\} = [\hat{f}(\omega)]^*$$

If we now use the convolution theorem with $\hat{g}(\omega) = [\hat{f}(\omega)]^*$, we have

$$\mathcal{F}\{f(x) * f(-x)\} = \hat{f}(\omega)[\hat{f}(\omega)]^* = \left|\hat{f}(\omega)\right|^2$$

Using the inverse transform

$$f(x) * f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 e^{-i\omega x}\,\mathrm{d}\omega$$

The LHS (letting $u = -s$) is

$$\int_{-\infty}^{\infty} f(u-s)f(-u)\,\mathrm{d}u = \int_{-\infty}^{\infty} f(x+s)f(s)\,\mathrm{d}s$$

In particular, setting $x = 0$, we obtain

$$\int_{-\infty}^{\infty}[f(u)]^2\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2\,\mathrm{d}\omega$$

which is the required result. ■

## 3.4 The Dirac delta-function (impulse function)

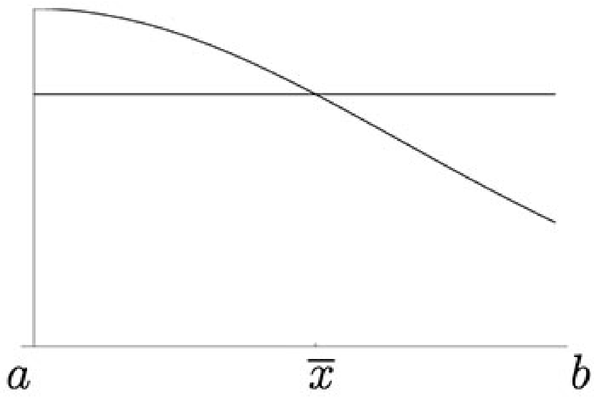
Before we define this we need to be aware of the following theorem:

**Theorem 3.6: Mean Value for Integrals**

If $f(x)$ is continuous on $[a, b]$ then

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(\bar{x})$$

for at least one $\bar{x}$ with $a \leq \bar{x} \leq b$.

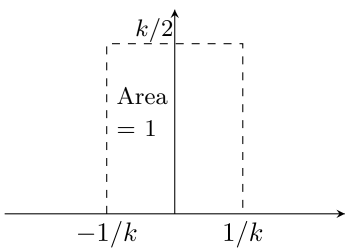



The proof follows from t
$f = F'$. Geometrically t
to that of a rectangle w

Now consider the following step function:

$$f_k(x) = \begin{cases} k/2, & |x| < 1/k, \\ 0, & |x| > 1/k \end{cases}$$

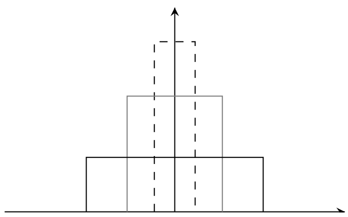
The function is sketched below:




Clearly we can see that an important property of the function is that

$$\int_{-\infty}^{\infty} f_k(x)\ \mathrm{d}x = \left(\frac{k}{2}\right)\left(\frac{2}{k}\right) = 1$$

As $k$ increases, $f_k(x)$ gets taller and thinner:



**Definition** (Dirac Delta function)**.**

$$\delta(x) = \lim_{k\to\infty} f_k(x)$$

although, of course, this limit doesn't exist in the usual mathematical sense. Effectively $\delta(x)$ is infinite at $x = 0$ and zero at all other values of $x$. The key property however, is that

$$\int_{-\infty}^{\infty} \delta(x)\ \mathrm{d}x = 1$$

The delta function is most useful in how it interacts with other functions.

## Sifting property of the delta function

Consider

$$\int_{-\infty}^{\infty} f(x)\delta(x)\ \mathrm{d}x$$

where $f$ is a continuous function over $(-\infty, \infty)$. Using our definition of the delta-function we can rewrite this as

$$\lim_{k\to\infty} \int_{-1/k}^{1/k} \frac{k}{2} f(x)\ \mathrm{d}x = \lim_{k\to\infty} f(\bar{x})\frac{k}{2}\left(\frac{1}{k} - \left(-\frac{1}{k}\right)\right)$$

for some $\bar{x}$ in $[-1/k, 1/k]$ using the mean-value theorem for integrals. Clearly as $k \to \infty$, we must have $\bar{x} \to 0$. Then the expression above simplifies to

$$\lim_{k\to\infty} f(0)\frac{k}{2} \cdot \frac{2}{k} = f(0)$$

We have therefore established that for any continuous function $f$:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\ \mathrm{d}x = f(0)$$

This result can easily be generalised to

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\ \mathrm{d}x = f(a)$$

**Example 3.7.** *Find the Fourier transform of $\delta(x)$.*

We have

$$\begin{aligned}\mathcal{F}\{\delta(x)\} &= \int_{-\infty}^{\infty} \delta(x)e^{-i\omega x}\ \mathrm{d}x \\ &= e^{-i\omega 0} = 1\end{aligned}$$

using the sifting property. From this we can deduce that the inverse Fourier transform of 1 is $\delta(x)$.

From this last result, and using the inversion formula, we see that an alternative representation of the delta function is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1)e^{\pm i\omega x}\ \mathrm{d}\omega$$

with the $\pm$ arising from the observation that $\delta(x)$ is an even function of $x$ about $x = 0$. If we interchange the variables we can also write

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\omega x}\ \mathrm{d}x$$

If we are prepared to work in terms of delta-functions, we can now take the Fourier transforms of function that do not decay as $x \to \pm\infty$.

**Example 3.8.** *Find the Fourier transform of* $\cos\omega_0 x$.

$$\begin{aligned}\mathcal{F}\{\cos\omega_0 x\} &= \int_{-\infty}^{\infty} \frac{1}{2}(e^{i\omega_0 x} + e^{-i\omega_0 x})e^{-i\omega x}\ \mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)x}\ \mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)x}\ \mathrm{d}x \\ &= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\end{aligned}$$

which is a two-spiked 'function'

*Exercise:* Find the

$i\pi\delta(\omega+\omega_0)$.


*"We can do this the easy way or we can do this the cute way."*

# 4 Partial Differential Equations

In this section we will consider second order partial differential equations (PDEs). The three main equations we will study are Lecture 25

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \qquad \textbf{Wave equation}$$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \qquad \textbf{Heat equation}$$

$$\nabla^2 \Phi = f(\mathbf{r}), \qquad \textbf{Laplace/Poisson equation}$$

## 4.1 The Wave Equation

For this discussion we will restrict ourselves to the one-dimensional version of the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

**Solution on finite domain in $x$: method of separation of variables**

Consider the following problem. Solve (1) for $u(x,t)$ subject to the boundary conditions:

$$u(0,t) = 0 \quad u(L,t) = 0 \quad (t \geq 0) \tag{2}$$

(string clamped at the end)

and the initial conditions

$$u(x,0) = f(x) \quad (0 \leq x \leq L) \tag{3}$$

$$\frac{\partial u}{\partial t} = 0 \quad (0 \leq x \leq L) \tag{4}$$

(released from rest). We see a solution to (1) of the 'separated-variables' form

$$u(x,t) = X(x)T(t) \quad (T \text{ is } \textit{not} \text{ tension})$$

(boundary condition's $X(0) = X(L) = 0$ to satisfy (2) initial condition (4) $\implies T'(0) = 0$).

Substituting the expression into (1) we get

$$XT''(t) = c^2 X''(x)T(t)$$

Dividing both sides by $c^2XT$, we can write this a s

$$\text{only depends on } t \left\{ \frac{T''}{c^2T} = \frac{X''}{X} \right\} \text{only depends on } x$$

We observe that the left-hand-side above only depends on $t$, while the right-hand-side is dependent only on $x$. The expression must hold for all $t > 0$ and $x \in [0, L]$. The only way this can be satisfied is if the left and right hand sides are equal to a constant.

In other words we must have

$$\frac{X''}{X} = K = \frac{T''}{c^2 T}$$

Where $K$ is the *separation constant.*

We have therefore reduced our PDE to two second order ordinary differential equations which should be easier to solve, particularly as they have constant coefficients. Now lets consider applying the boundary conditions. These imply that

$$X(0) = X(L) = 0$$

with $X(x)$ satisfying

$$X'' - KX = 0 \tag{5}$$

The form of the general solution depends on whether $K$ is positive, zero or negative. Lets consider those cases in turn.

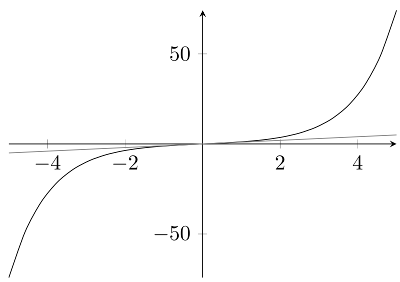
(i) $K > 0$. In this case, the general solution of (5) is

$$X = A\cosh(\sqrt{K}x) + B\sinh(\sqrt{K}x)$$

However if we then apply the boundary conditions, we have

$$\begin{aligned} X(0) = 0 &\implies A = 0 \\ X(L) = 0 &\implies B\sinh(\sqrt{K}L) = 0 \implies B = 0 \text{ or } \sinh(\sqrt{K}L) = 0 \end{aligned}$$

Neither of these options are acceptable: the first leads to $X$ identically zero, while a consideration of the graph of $\sinh x$ versus $x$ shows that no such positive value for $K$ exists.




We therefore conclude that for these boundary conditions the constant $K$ cannot be positive.

(ii) $K = 0$. In this case the solution of (5) is

$$X = ax + b$$

Again, if we apply $X(0) = X(L) = 0$, we see that no non-zero solution is possible.

(iii) $K < 0$. It is convenient to set $K = -\lambda^2$. Then (5) becomes $X'' + \lambda^2 X = 0$, and the general solution is

$$X = A\cos\lambda x + B\sin\lambda x$$

Applying the boundary conditions $X(0) = 0 \implies A = 0$ as before, but now the condition $X(L) = 0$ leads to $\sin(\lambda L) = 0$ for which there are an infinite number of solutions

$$\lambda = n\pi/L, \quad n = \pm 1, \pm 2, \ldots \tag{6}$$

The solution for $X$ is therefore

$$X = B_n \sin(n\pi x/L)$$

Now we turn to the corresponding equation for $T$ which takes the form $T'' + c^2\lambda^2 T = 0$ and therefore has the general solution

$$T = C\cos(\lambda ct) + D\sin(\lambda ct)$$

The initial condition (4) implies that $T'(0) = 0$ and this leads to $D = 0$. Substituting for $\lambda$ from (6) we are left with $T = C_n \cos(n\pi ct/L)$. A solution that satisfies the wave equation and conditions (2), (4) is therefore

$$y_n = XT = \beta_n \sin(n\pi x/L)\cos(n\pi ct/L)$$

where we have introduced $\beta_n = B_n C_n$. Since the wave equation is linear it follows that any linear combination of the solutions for different $n$ is also a solution. The most general solution is therefore of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \beta_n \sin(n\pi x/L)\cos(n\pi ct/L)$$

which can be expressed more succinctly as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\cos(n\pi ct/L) \tag{7}$$

where we have written $b_n = \beta_n - \beta_{-n}$. We have one more initial condition to apply - the condition that $u = f(x)$ when $t = 0$. Imposing this, we see that $f(x)$ is related to the unknown coefficients $b_n$ in the following way:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L), \quad (0 < x < L) \tag{8}$$

We recognise this as a half-range Fourier sine series for $f(x)$. Our Fourier series studies of §2 have shown us that

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,\mathrm{d}x$$

and so can be computed for a given $f(x)$. The required solution to the wave equation is therefore given by the infinite sum (7), with the coefficients $b_n$ calculated from (8).

*Exercise:* Solve the wave equation subject to

$$u(0,t) = u(L,t) \quad \text{for } t \geq 0$$
$$u(x,0) = 0 \quad \text{for } 0 \leq x \leq L$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for } 0 \leq x \leq L$$

Answer:

$$u = \sum_{n=1}^{\infty} \beta_n \sin(n\pi x/L)\sin(n\pi ct/L)$$

where

$$\beta_n = \frac{2}{2\pi c}\int_0^L g(x)\sin(n\pi x/L)\ \mathrm{d}x$$

**Solution on infinite domain: Fourier transforms**

Suppose we have to solve the following problem: Lecture 26

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty,\ t > 0$$

$$u(x,0) = 4e^{-5|x|} \quad \text{for } -\infty < x < \infty \tag{9}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \tag{10}$$

with $u$ bounded as $x \to \pm\infty$ for all $t$.

We take the Fourier transform in $x$ of the differential equation, using property (vii) from section 3.2. This gives

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\omega^2\hat{u}$$

where $\hat{u}(\omega,t)$ is the Fourier transform of $u(x,t)$. The general solution is

$$\hat{u}(\omega,t) = A(\omega)\cos(\omega ct) + B(\omega)\sin(\omega ct)$$

The initial condition (10) implies that $\partial\hat{u}/\partial t = 0$ at $t = 0$, and so we conclude that $B(\omega) = 0$. Then applying condition (9) we see that

$$A(\omega) = \mathcal{F}\{4e^{-5|x|}\} = \frac{40}{25+\omega^2}$$

using a result we saw on Problem Sheet 6. Hence we have

$$\hat{u}(\omega,t) = \frac{40}{25+\omega^2}\cos(\omega ct)$$

We can invert this using the convolution theorem, since $\hat{u}$ is the product of two terms we know the individual inverses of. We proceed as follows.

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left\{\frac{40}{25+\omega^2}\cos(\omega ct)\right\} \\ &= \mathcal{F}^{-1}\left\{\frac{40}{25+\omega^2}\right\} * \mathcal{F}^{-1}\left\{\cos(\omega ct)\right\} \\ &= 4e^{-5|x|} * \mathcal{F}^{-1}\{\cos(\omega ct)\} \end{aligned} \tag{11}$$

Now to find the inverse transform of the second term we can use the result from section 3.5.3 that

$$\mathcal{F}\{\cos(\omega_0 x)\} = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

where $\delta$ is the Dirac delta function. Then using the symmetry formula (property (vi) in section 3.2: $\mathcal{F}[\hat{F}(x)\} = 2\pi F(-\omega)$ it follows that

$$\mathcal{F}\{\pi\delta(x+\omega_0) + \pi\delta(x-\omega_0)\} = 2\pi\cos(-\omega_0\omega)$$

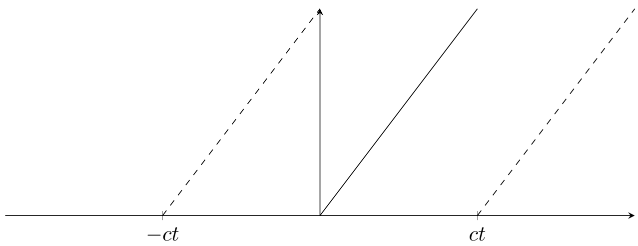
and hence

$$\mathcal{F}^{-1}\{\cos(\omega_0\omega)\} = \frac{1}{2}\delta(x+\omega_0) + \frac{1}{2}\delta(x-\omega_0)$$

Using this result in (11):

$$\begin{aligned}
u(x,t) &= 4e^{-5|x|} * \frac{1}{2}(\delta(x+ct) + \delta(x-ct)) \\
&= 2\int_{-\infty}^{\infty} e^{-5|x-s|}\left\{\delta(s+ct) + \delta(s-ct)\right\}\,\mathrm{d}s \\
&= 2e^{-5|x+ct|} + 2e^{-5|x-ct|}
\end{aligned}$$

with the last line following from the sifting property of the delta function (section 3.5.3).




## D'Alembert's solution for the wave equation

The particular solution we obtained by separation of variables can be rewritten as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}(x+ct)\right) + \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}(x-ct)\right)$$

Similarly, the solution we obtained by Fourier transforms also depends on the combination of variables $x - ct$ and $x + ct$. The functional dependence on these quantities indicates that in both cases the solution is the sum of a left-travelling $(x + ct)$ and right-travelling $(x - ct)$ wave, with both waves propagating at speed c. This observation provides us with some motivation for the following study which results in the derivation of the general solution of the wave equation.

We introduce new variables

$$\xi = x + ct, \quad \eta = x - ct$$

The partial derivatives transform as follows:

$$\left.\frac{\partial}{\partial x}\right|_t = \left.\frac{\partial \xi}{\partial x}\right|_t \frac{\partial}{\partial \xi} + \left.\frac{\partial \eta}{\partial x}\right|_t \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\left.\frac{\partial}{\partial t}\right|_x = \left.\frac{\partial \xi}{\partial t}\right|_x \frac{\partial}{\partial \xi} + \left.\frac{\partial \eta}{\partial t}\right|_x \frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

We can then calculate

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$
$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(x\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}\right)$$
$$= c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}$$

Under this transformation the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

becomes

$$-4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

The equation is in its *canonical form*.

This equation can be integrated once with respect to $\xi$ to give

$$\frac{\partial u}{\partial \eta} = f'(\eta)$$

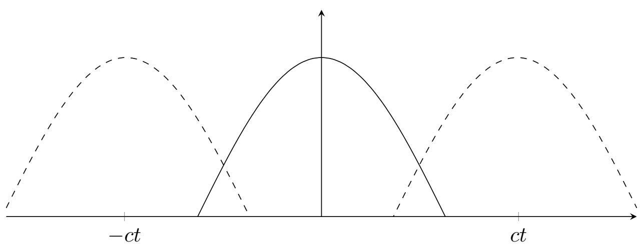
where $f'$ is an arbitrary function of $\eta$. Integrating again, this time with respect to $\eta$, we obtain

$$u = f(\eta) + g(\xi)$$

with $g$ an arbitrary function of $\eta$. The general solution of the wave equation therefore has the form

$$u = f(x - ct) + g(x + ct)$$

and so can always be written as the sum of right and left travelling waves.




## 4.2 The Heat Equation

In one dimension this is the partial differential equation Lecture 27

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \tag{12}$$

Again we will look at some straightforward methods of solution for simple geometries.

**Solution on a finite domain: separation of variables**

Suppose we have boundary conditions of the form

$$u(0,t) = u(L,t) = 0 \text{ for } t > 0 \tag{13}$$

and an initial condition

$$u(x,0) = f(x) \text{ for } 0 \le x \le L \tag{14}$$

In a similar way to the wave equation we can seek a separated-variables solution of the form

$$u(x,t) = X(x) + T(t)$$

Substitution into (12) leads to

$$XT' = \kappa X''T$$

so that

$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda^2$$

($-\lambda^2$ is the separation constant.) As in the case of the wave equation we now see that the left-hand-side is a function of $x$ only, and the right-hand-side depends only on $t$. Therefore we must have, for some constant $\lambda^2$:

$$X'' + \lambda^2 X = 0; \quad T' + \lambda^2 \kappa T = 0$$

The sign of the separation constant $\lambda^2$ depends on the type of boundary conditions we impose. In our particular case since we are imposing (13) we require $X(0) = X(L) = 0$ which can only be accomplished if $\lambda^2 > 0$ (otherwise the solutions for $X$ will be of exponential form). We therefore have

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

with $A = 0$ and $\lambda = n\pi/L$. The corresponding solution for $T$ arises to

$$\frac{T'}{T} = -\lambda^2\kappa = -n^2\pi^2\kappa/L^2$$

and hence

$$T(t) = c\exp\left(\frac{-n^2\pi^2\kappa t}{L^2}\right)$$

Putting the two components together and summing over all modes we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-n^2\pi^2\kappa t}{L^2}\right) \tag{15}$$

Finally, applying the initial condition (14) we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (0 < x < L)$$

which we recognise as a half-range Fourier sine series for $f(x)$ with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \tag{16}$$

Therefore the required solution of the heat equation subject to (13) and (14) is given by (15) and (16). The method can be adapted to accommodate boundary conditions on $\partial u/\partial x$ rather than $u$ [see problem sheet 7].

**Solutions on an infinite or semi-infinite domain**

Again, as with the wave equation we can use Fourier transforms to help us obtain a solution. Consider for example the problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \quad \text{for } \ 0 < x < \infty,\ t > 0$$

$$u(0,t) = 0 \quad \text{for } \ t \geq 0 \tag{17}$$

$$u(x,0) = f(x) \quad \text{for } \ 0 < x < \infty \tag{18}$$

and $u$ bounded as $x \to \infty$ for all $t$.

Since this problem is posed over a semi-infinite domain we could take either a Fourier cosine or sine transform. Recall from property (ix), section 3.2 that if we take a cosine transform of a second derivative we require a knowledge of $\partial u/\partial x$ when $x = 0$, while if we take a sine transform we need to know $u$ at $x = 0$. In this particular case, in view of (17), we have the latter situation and so we will take a Fourier sine transform of the equation, to give

$$\frac{\partial \hat{u}_s}{\partial x} = \kappa\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial u^2}\right\} = \kappa\Big\{-\omega^2\hat{u}_s + \omega \underset{\underset{0}{\parallel}}{u(0,t)}\Big\}$$

where $\hat{u}_s(\omega,t)$ is the Fourier sine transform of $u(x,t)$ with respect to $x$. Substituting for $u(0,t)$ fro (17) and integrating, we obtain

$$\hat{u}_s = B(\omega)e^{-\omega^2\kappa t}$$

Taking the Fourier since transform of (18) we obtain $\hat{u}_s = \hat{f}_s(\omega)$ on $t = 0$, allowing us to determine $B(\omega) = \hat{f}_s(\omega)$ and hence

$$\hat{u}_s = \hat{f}_s(\omega)e^{-\omega^2\kappa t}$$

Applying the inversion formula for the Fourier sine transform we can them write the solution in the form

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(\omega)e^{-\omega^2\kappa t}\sin(\omega x)\,\mathrm{d}\omega$$

where

$$\hat{f}_s(\omega) = \int_0^\infty f(q)\sin(\omega q)\,\mathrm{d}q$$

**Higher dimensions**

The methods of separation of variables and Fourier transforms can also be used on the heat equation in two and three dimensions. However the variables in the boundary conditions need to be separated for this technique to work, and so the methods are of limited use for problems with complicated geometries

## 4.3 Laplace's equation and Poisson's equation

We will write Laplace's equation in the form

$$\nabla^2 \phi = 0$$

We will also study the related equation

$$\nabla^2 \phi = f(\mathbf{r}) \tag{19}$$

where $f$ is a prescribed function of position $\mathbf{r}(= x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$. Equation (19) is known as *Poisson's equation.* Note that these are both steady equations.

**Types of boundary conditions**

If we look for a solution in a volume $V$, then the boundary conditions will be given on the surfaces $S$ which bound $V$. These boundary conditions are generally of two types:

(i) *Dirichlet boundary conditions*, in which $\phi$ is given on the boundary;

(ii) *Neumann boundary conditions*, in which the normal derivative $\partial\phi/\partial n$, is prescribed on the boundary.

**Solution using separation of variables and transform techniques**

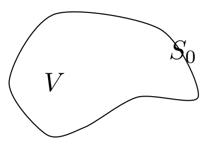
The same techniques used for the wave and heat equations will also work on Laplaces equation. The main limitations of these methods are the simple shapes of domain to which they can be applied, and the necessity for the variables in the boundary conditions to separate. There are examples of the use of these techniques on problem sheets 7 and 8.

In what follows we shall consider the three-dimensional form of the equations and apply some of the techniques and theorems we have learned to formulate solutions. We will use a number of results from vector calculus and will also generalise the Dirac delta function we encountered in section 3.

**Proposition 4.1** (Uniqueness for interior problems)**.** *Let $\phi$ satisfy Poisson's equation*

$$\nabla^2 \phi = f(\mathbf{r})$$

*in a volume $V$. The volume is bounded on its exterior by a surface $S_0$. On the surface we have a Dirichlet boundary condition, i.e. $\phi = p(\mathbf{r})$ on $S_0$. Then the solution for $\phi$ is unique.*




*Proof.* We will suppose there are two solutions and seek a contradiction. Let the two solutions be $\phi_1$ and $\phi_2$. They must both satisfy Poissons equation and the same boundary conditions. Forming the difference

$$\Phi \equiv \phi_1 - \phi_2$$

we must therefore have that

$$\nabla^2\Phi = \nabla^2\phi_1 - \nabla^2\phi_2 = f(\mathbf{r}) - f(\mathbf{r}) = 0$$

Now recall from 1.8 Green's first identity with $\psi = \phi = \Phi$:

$$\int_{S_0} \Phi \frac{\partial \Phi}{\partial n}\, \mathrm{d}S = \int_V \Phi \underbrace{\nabla^2\Phi}_{0} + |\nabla\Phi|^2 \, \mathrm{d}V$$

In view of the boundary conditions, ($\Phi = 0$ on $S_0$, so) the left-hand side is zero, and since $\Phi$ satisfies Laplaces equation throughout $V$, the first term on the right-hand-side is also zero. This leaves

$$\int_V |\nabla\Phi|^2 \, \mathrm{d}V = 0$$

The volume integral of a positive quantity can only be zero if the integrand is in fact identically zero, and so this implies

$$\nabla\Phi = 0 \text{ throughout } V$$

This means that $\Phi$ is at most a constant throughout $V$, but because $\Phi$ is zero on the boundaries, it follows that $\Phi$ is identically zero throughout $V$. Hence $\phi_1 = \phi_2$ and the solution is unique. ∎

Much the same reasoning applies if $\phi$ is a complex-valued function of position, if we have Neumann rather than Dirichlet boundary conditions, and also if the volume $V$ has holes in it [Problem sheet 8].

**Example 4.2.** Solve $\nabla^2\Phi = 2$ inside the unit sphere $r \leq 1$ with $\Phi = 1$ on $r = 1$. Lecture 28

We note that the right-hand-side, the geometry and the boundary conditions have radial symmetry. We therefore seek a solution $\Phi = \Phi(r)$ independent of $\theta, \phi$ in spherical polar coordinates. Referring back to section 1.9.10, the form for the Laplacian in spherical polars is

$$\nabla^2\Phi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \Phi}{\partial \phi}\right)\right]$$

$$= \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \cancel{\frac{\cot\theta}{r^2}\frac{\partial \Phi}{\partial \theta}} + \cancel{\frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2}} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial \phi^2}}$$

so that our equation reduces under the conditions of radial symmetry to

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{d\Phi}{\mathrm{d}r}\right) = 2$$

Integrating to obtain

$$r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{2r^3}{3} + C$$

and hence

$$\Phi = \frac{1}{3}r^2 - \frac{C}{r} + D$$

For $\Phi$ to be finite at $r = 0$ we require $C = 0$. Applying $\Phi = 1$ on $r = 1$ gives $D = 2/4$ and so the required solution is

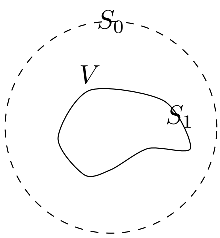
$$\Phi = \frac{1}{3}r^2 + \frac{2}{3}$$

Due to the uniqueness theorem, we know this is the only possible solution.

**Proposition 4.3** (Uniqueness for exterior problems)**.** *Now suppose we have a situation where there is an inner boundary $S_1$, but the outer boundary $S_0$ is taken to infinity so that $V$ is now an unbounded volume (figure 3). Suppose*

$$\nabla^2 \phi = f(\mathbf{r})$$

*throughout $V$. Suppose in addition that $\phi = \mathcal{O}(1/r)$, $\partial\phi/\partial r = \mathcal{O}(1/r^2)$ as $r \to \infty$. Then the solution for $\phi$ in $V$ is unique.*[1]




*Proof.* Let's start by considering the surface $S_0$ to be a large sphere of radius $R$. Suppose there are two solutions $\phi_1, \phi_2$ and form the difference $\Phi$. Proceeding as in the previous proof, using Green's first identity, we have

$$\begin{aligned}\int_V |\nabla\Phi|^2 \, \mathrm{d}V &= \sum_{i=0}^{1} \int_{S_i} \Phi \frac{\partial \Phi}{\partial n} \, \mathrm{d}S \\ &= \int_{S_0} \Phi \frac{\partial \Phi}{\partial n} \, \mathrm{d}S\end{aligned}$$

since the integral over $S_1$ is zero due to the boundary condition that $\Phi$ vanishes on $S_1$. Since $S_0$ is a sphere of radius $R$, we can write $\mathrm{d}S = R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ in spherical polar coordinates, and $\partial\Phi/\partial n = \partial\Phi/\partial r$. We therefore have

$$\int_V |\nabla\Phi|^2 \, \mathrm{d}V = \int_{\phi=0}^{2\pi} \int_0^{\pi} \Phi \frac{\partial \Phi}{\partial r} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

Because of the assumed behaviour of $\Phi$ as $r \to \infty$, the RHS above is of order $1/R$ and hence tends to zero as $R \to \infty$. We therefore see that the exterior problem

$$\int_V |\nabla\Phi|^2 \, \mathrm{d}V = 0$$

and hence, arguing as in the previous proof, $\Phi = 0$ and hence the solution is unique. ■

[1] If $f = \mathcal{O}(r^\alpha)$ as $r \to \infty$, this means that $\lim_{r\to\infty} r^{-\alpha} f(r) = K \neq 0$.

The proof extends to Neumann boundary conditions as before, and also to the situation where the volume $V$ has any finite number of inner boundaries [Problem sheet 8]

**Example 4.4.** Solve $\nabla^2\Phi = f(r)$ for $0 \leq r < \infty$, where

$$f(r) = \begin{cases} f_0, & r \leq a \\ 0, & r > a \end{cases}$$

We will assume that $\Phi$ is bounded throughout the region and that $\Phi$ and $\mathrm{d}\Phi/\mathrm{d}r \to 0$ as $r \to \infty$ to guarantee a unique solution. As before we can seek a solution with radial symmetry and so $\Phi = \Phi(r)$ only.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = f(r) \tag{20}$$

Solving for $r \leq a$ first, we obtain (after some elementary calculus):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = f_0 r^2 \implies \Phi = 16 f_0 r^2 - \cancel{\frac{A}{r}} + B$$

We need $A = 0$ so that the solution is finite at $r = 0$. Next, solving (20) for $r > a$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 0 \implies \Phi = \frac{C}{r} + D$$

We require $\Phi \to 0$ as $r \to \infty$, and so we deduce that $D$ must be zero. To find the remaining constants $B, C$ we impose continuity of $\Phi$ and $\mathrm{d}\Phi/\mathrm{d}r$ at $r = a$. This gives

$$\text{continuity of } \Phi \implies \frac{1}{6}f_0 a^2 + B = \frac{C}{a}$$

$$\text{continuity of } \frac{\mathrm{d}\Phi}{\mathrm{d}r} \implies \frac{1}{3}f_0 a = -\frac{C}{a^2}$$

Substituting for $B$ and $C$ we find that the resulting solution for $\Phi$ is

$$\Phi = \begin{cases} \dfrac{1}{6}f_0 r^2 - \frac{1}{2}f_0 a^2, & r \leq a \\ -\dfrac{1}{3}f_0 a^3/r, & r \geq a \end{cases}$$

## Point sources and the Dirac delta function

Lets consider the result of our previous example in the limit as $a \to 0$. This means that the right-hand side is becoming concentrated at the origin $r = 0$. In addition we will let $f_0 \to \infty$ in such a way that $(4/3)\pi a^3 f_0$ remains equal to a constant - call this $K$. In this limit the solution we obtained above for $\Phi$ for $r > a$ becomes

$$\Phi = -\frac{K}{4\pi r}$$

We have therefore obtained a solution to

$$\nabla^2\Phi = f(r)$$

where the RHS has the properties

$$f(r) = \begin{cases} 0 & r \neq 0 \\ \text{“}\infty\text{”} & r = 0 \end{cases}$$

Another property of the function $f(r)$ can be seen from the following calculation, in which we use the divergence theorem over a sphere radius $R$ centred at the origin

$$\begin{aligned}\int_V f(r)\ \mathrm{d}V &= \int_V \nabla^2\Phi\ \mathrm{d}V \\ &= \int_V \nabla\cdot(\nabla\Phi)\ \mathrm{d}V \\ &= \int_{\text{sphere } r=R} \left.\frac{\partial\Phi}{\partial r}\right|_{r=R} \mathrm{d}S \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{K}{4\pi R^2} R^2\sin\theta\ \mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{K}{4\pi}2\pi[-\cos\theta]_0^\pi \\ &= K\end{aligned}$$

The function $f/K$ is the extension of the Dirac delta function studied in §3 to three dimensions and is denoted by $\delta(r)$. Moreover, we can extend this definition so that the argument is a vector, i.e. the function $\delta(\mathbf{r})$. Lecture 29

**Definition** (Delta function with vector argument)**.** We define

$$\delta(\mathbf{r}) = 0 \text{ for } \mathbf{r} \neq 0$$

and

$$\int_V \delta(\mathbf{r})\ \mathrm{d}V = \begin{cases} 1 & \text{if } V \text{ contains the origin} \\ 0 & \text{otherwise} \end{cases}$$

We claim that the solution of

$$\nabla^2\Phi = K\delta(\mathbf{r}) \tag{21}$$

is

$$\Phi = -\frac{K}{4\pi|\mathbf{r}|} \tag{22}$$

This can be verified in the following way . Firstly we have already seen in section 1.4.6 that $\nabla^2(1/r) = 0$ for $r \neq 0$. To check the solution for $r = 0$ we integrate both sides of (21) over a volume $V$ containing $r = 0$ to get

$$\int_V \nabla^2\Phi\ \mathrm{d}V = K\int_V \delta(\mathbf{r})\ \mathrm{d}V = K$$

Using the divergence theorem, the LHS can be rewritten as

$$\int_S (\nabla\Phi \cdot \hat{\mathbf{n}})\ \mathrm{d}S$$

where the origin is interior to the closed surface $S$ which bounds $V$. But if $\Phi = -K/4\pi|\mathbf{r}|$, then

$$\nabla\Phi = -\frac{K}{4\pi}\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{K}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} = \frac{K}{4\pi}\frac{\mathbf{r}}{r^3}$$

and so this can be rewritten as

$$\frac{K}{4\pi}\int_S \frac{\mathbf{r}}{r^3}\cdot\hat{\mathbf{n}}\ \mathrm{d}S = \begin{cases} \dfrac{K}{4\pi}\cdot 4\pi = K & \text{if 0 is inside } V \\ 0 & \text{if 0 is outside} \end{cases}$$

The integral here should be familiar as it is equal to $4\pi$ using Gauss' flux theorem (Theorem 1.22). The LHS is therefore also equal to $K$, and we have therefore verified that (22) is indeed the solution to (21), and is the only solution since we can invoke the uniqueness theorem.

If we move the source to $\mathbf{r} = \mathbf{r}_0$ then we can easily modify our analysis to show that the solution of $\nabla^2\Phi = K\delta(\mathbf{r} - \mathbf{r}_0)$ is

$$\Phi = -\frac{K}{4\pi|\mathbf{r} - \mathbf{r}_0|}$$

It can also be shown that the sifting property of the delta function carries over to three dimensions, i.e.

$$\int_V g(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_1)\ \mathrm{d}V = g(\mathbf{r}_1)$$

for any continuous function $g$.

**Definition** (Green's function)**.** The Green's function $G(\mathbf{r};\mathbf{r}_0)$ is defined as the solution of

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \tag{23}$$

subject to some appropriate boundary conditions.

For the three-dimensional problems we have studied we have seen that the so-called free-space Greens function, i.e. the function that satisfies (23) and tends to zero as $r \to \infty$ is given by

$$\begin{aligned} G(\mathbf{r};\mathbf{r}_0) &= -\frac{1/4\pi}{|\mathbf{r} - \mathbf{r}_0|} \\ &= -(4\pi)^{-1}\left\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right\}^{-1/2} \end{aligned}$$

in Cartesian coordinates. As we shall see, knowledge of the Greens function will enable us to write down solutions to Poissons and Laplaces equation in closed form.

## Solutions to Poisson's equation using Green's functions

Suppose that $\Phi$ satisfies

$$\nabla^2\Phi = f(\mathbf{r})$$

throughout some volume $V$. We will denote the boundary of $V$ by the surface $\partial V$. [N.B. not $\mathrm{d}V$]. (If the volume is unbounded we will assume that $\Phi = \mathcal{O}(1/r)$ as $r \to \infty$, so that a unique solution is guaranteed). We suppose that the boundary condition is of Dirichlet-type, i.e.

$$\Phi = p(\mathbf{r}) \text{ on } \partial V$$

Consider the following associated problem for $G(\mathbf{r};\mathbf{r}_0)$:

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$$

in $V$ with $G = 0$ on $\partial V$. Now take Green's second identity (1.8.8) and apply it to the function $\Phi$ and $G$:

$$\int_V (\Phi\nabla^2 G - G\nabla^2\Phi)\ \mathrm{d}V = \int_{\partial V} \left(\Phi\frac{\partial G}{\partial n} - G\frac{\partial \Phi}{\partial n}\right)\ \mathrm{d}S$$

The right hand side can be rewritten as

$$\int_{\partial S} p(\mathbf{r})\frac{\partial G}{\partial n}(\mathbf{r};\mathbf{r}_0)\ \mathrm{d}S$$

while after substituting for $\nabla^2 G$ and $\nabla^2\Phi$, the LHS is

$$\int_V \Phi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0) - G(\mathbf{r};\mathbf{r}_0)f(\mathbf{r})\ \mathrm{d}V$$

which, upon use of the sifting property, becomes

$$\Phi(\mathbf{r}_0) - \int_V G(\mathbf{r};\mathbf{r}_0)f(\mathbf{r})\ \mathrm{d}V$$

We can therefore write the solution for $\Phi$ in the form

$$\Phi(\mathbf{r}_0) = \int_V G(\mathbf{r};\mathbf{r}_0)f(\mathbf{r})\ \mathrm{d}V + \int_{\partial V} p(\mathbf{r})\frac{\partial G}{\partial n}(\mathbf{r};\mathbf{r}_0)\ \mathrm{d}S \tag{24}$$

Thus, in principle, if we can find the Green's function we can solve Poisson's equation for $\Phi$.

A similar approach can be taken if Poisson's equation is subject to Neumann conditions on the boundary [Problem sheet 8].

## The method of images

Often it is possible to find the Greens function explicitly by the method of images as shown in the following Laplace equation example.

Suppose we wish to solve

$$\nabla^2\Phi = 0$$

in the region $z > 0$ with the boundary condition

$$\Phi(x, y, 0) = p(x, y)$$

(We will assume $\Phi = \mathcal{O}(1/r)$ as $z \to \infty$ to ensure uniqueness). We will tackle this problem by introducing an associated *Dirichlet Green's function* that satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \text{ for } z > 0 \tag{25}$$

$$G = 0 \text{ on } z = 0 \tag{26}$$

If the boundary condition at $z = 0$ is absent we know that the Greens function is

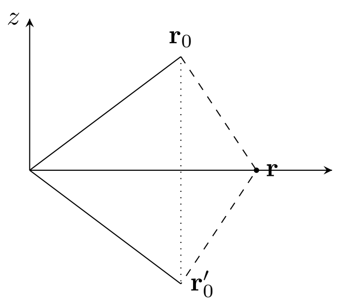
$$G(\mathbf{r};\mathbf{r}_0) = -(4\pi)^{-1}\frac{1}{|\mathbf{r} - \mathbf{r}_0|}$$

This solution is referred to as a source singularity of strength 1 situated at $\mathbf{r} = \mathbf{r}_0 = (x_0, y_0, z_0)$. Now lets add another singularity of opposite strength at a location the same distance below the $xy$ plane, i.e. at $\mathbf{r}_0' = (x_0, y_0, -z_0)$ (a mirror image). The modified Greens function is therefore

$$G(\mathbf{r};\mathbf{r}_0) = -(4\pi)^{-1}\frac{1}{|\mathbf{r} - \mathbf{r}_0|} + (4\pi)^{-1}\frac{1}{|\mathbf{r} - \mathbf{r}_0'|}$$

Now, when $z = 0$ we have

$$|\mathbf{r} - \mathbf{r}_0| = |\mathbf{r} - \mathbf{r}_0'|$$




and so we see that $G = 0$ on $z = 0$ as required. Thus the Greens function (27) satisfies equation (25) and the boundary condition (26)

Now that we have the Green's function we can apply the result of the previous section (equation (24) with $f = 0$) to obtain the solution for $\Phi$ as Lecture 30

$$\Phi(\mathbf{r}_0) = \int_{\partial V} p(x, y)\frac{\partial G}{\partial n}(\mathbf{r};\mathbf{r}_0)\,\mathrm{d}S \quad (f = 0)$$

In this example $\partial V$ is the plane $z = 0$ and $\partial/\partial n = -\partial/\partial z$ (since $n$ is the outward normal to the volume $V$). Using our expression for $G$

$$G = -(4\pi)^{-1}\Big\{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-1/2} - [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{-1/2}\Big\}$$

we find that

$$\frac{\partial G}{\partial z} = (-4\pi)^{-1}\Big\{ -(z - z_0)[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-3/2} + (z + z_0)[(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{-3/2}\Big\}$$

Evaluating this quantity on $z = 0$ we have

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = -(4\pi)^{-1} 2z_0 \left[(x-x_0)^2 + (y-y_0)^2 + z_0^2\right]^{-3/2}$$

and so the solution for $\Phi$ can be expressed in the form

$$\Phi(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \left[(x-x_0)^2 + (y-y_0)^2 + z_0^2\right]^{-3/2} \mathrm{d}x\,\mathrm{d}y$$

**Example 4.5.** Suppose we wish to solve Laplace's equation for $z > 0$ with $\Phi = p(x, y)$ on $z = 0$ and $p(x, y)$ given explicitly as

$$p(x,y) = \begin{cases} 1, & x^2 + y^2 \leq 1 \\ 0, & x^2 + y^2 > 1 \end{cases}$$

Using the result derived above with this specific form for $p$:

$$\Phi(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{x^2+y^2\leq 1} \left[(x-x_0)^2 + (y-y_0)^2 + z_0^2\right]^{-3/2} \mathrm{d}x\,\mathrm{d}y$$

Using polar coordinates: $x = \rho\cos\theta$, $y = \rho\sin\theta$:

$$= \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^1 \left[\rho^2 + x_0^2 + y_0^2 + z_0^2 - 2x_0\rho\cos\theta - 2y_0\rho\sin\theta\right]^{-3/2} \rho\ \mathrm{d}\rho\,\mathrm{d}\theta$$

where we have switched to plane polar coordinates $(\rho, \theta)$. In particular, the solution along the $z$-axis is

$$\begin{aligned} \Phi(0,0,z_0) + \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^1 (\rho^2 + z_0^2)^{-3/2} \rho\ \mathrm{d}\rho\,\mathrm{d}\theta \\ = -z_0[(\rho^2 + z_0^2)^{-1/2}]_{\rho=0}^{\rho=1} \\ = 1 - \frac{z_0}{(1+z_0^2)^{1/2}} \end{aligned}$$

We can apply the method of images in a similar fashion to solve the same problem but with a Neumann condition on $z = 0$ [Problem sheet 8].

## Poisson's equation in two dimensions

**Felina.** This same approach can also be used for two-dimensional problems, although the Greens function has a different (logarithmic) form in this case [see problem sheet 8]. For a Dirichlet problem in which we wish to solve

$$\nabla^2 \phi = f(\mathbf{r})$$

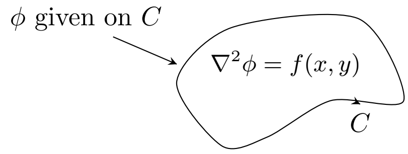
in a region $R$ of the $x - y$ plane, with

$$\phi = p(x)$$

on the boundary $C$ of $R$ (figure 5), we consider the Green's function problem

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$$

in $R$, with $G = 0$ on $C$.




Applying Green's second identity in 2D (section 1.8.9) we find that

$$\phi(\mathbf{r}_0) = \int_R G(\mathbf{r}; \mathbf{r}_0) f(\mathbf{r})\ \mathrm{d}x\,\mathrm{d}y + \oint_C p(x) \frac{\partial G}{\partial n}(\mathbf{r}; \mathbf{r}_0)\ \mathrm{d}s$$

where now $\mathbf{r}_0 = (x_0, y_0)$. A similar expression can be derived for Neumann boundary conditions. By using the method of images as we did in three dimensions, Greens functions can be found explicitly for certain problems. There is an example on the final problem sheet.

**- End of Multivariable Calculus -**